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ABSTRACT

These 14 research reports are grouped into three broad categories based on the Piagetian level concerned. The articles concerning preoperational concepts focus on problems such as: (1) finding an appropriate mathematical description of some of the primitive mathematical concepts; (2) the role of "activities" in early concept acquisition; (3) the similarities and differences between perceptual and conceptual processes; (4) the relation between memory improvement and improved operational ability; and (5) variables responsible for the large gap between visual-visual and haptic-visual discrimination. The articles concerning the transitional phases between concrete and formal operations focus on elementary transformation geometry concepts and emphasize the role that figurative content can play in influencing the difficulty of Piaget-type tasks. Two articles in this group focus on affine transformations, similarities, or projections. The articles dealing with older subjects or formal operational concepts investigate topics such as: (1) the nature of specific mathematical concepts as they relate to the cognitive structure of the learner; (2) understanding of frames of reference by preservice teacher education students as determined by Piagetian tasks; and (3) the bias for upright figures that students exhibit in forming concepts. (MP)

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RECENT RESEARCH CONCERNING THE DEVELOPMENT OF SPATIAL AND GEOMETRIC CONCEPTS

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and Environmental Education
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MATHEMATICS EDUCATION REPORTS

The Mathematics Education Reports series makes available recent analyses and syntheses of research and development efforts in mathematics education. As a part of this series, we are pleased to make available this collection of research reports in the area of spatial and geometric concepts. The thirteen studies reported here were coordinated by the Geometry Working Group of the Georgia Center for the Study of Learning and Teaching Mathematics.

Other Mathematics Education Reports make available information concerning mathematics education documents analyzed at the ERIC Information Analysis Center for Science, Mathematics, and Environmental Education. These reports fall into three broad categories. Research reviews summarize and analyze recent research in specific areas of mathematics education. Resource guides identify and analyze materials and references for use by mathematics teachers at all levels. Special bibliographies announce the availablility of documents and review the literature in selected interest areas of mathematics education.

Priorities for the development of future Mathematics Education Reports are established by the advisory board of the Center, in cooperation with the National Council of Teachers of Mathematics, the Special Interest Group for Research in Mathematics Education, and other professional groups in mathematics education. Individual comments on past Reports and suggestions for future Reports are always welcome.

Jon L. Higgins
Associate Director for
Mathematics Education

The editors would like to give special thanks to the following people who served as referees and reviewers for the articles appearing in this monograph, and as consultants when the projects were in the planning stages.

Alan Hoffer	Mathematics Department University of Oregon
Les Steffe	Department of Mathematics Education University of Georgia
Art Coxford	School of Mathematics Education University of Michigan
Paul Trafton	Department of Mathematics Education National College of Education

The editors would also like to thank the many other members of our space and geometry research group who attended planning conferences and completed projects that do not appear in this monograph. The editors believe that the existence of a successful, non-funded, multi-institutional, cooperative research project is in itself a significant research innovation in mathematics education.

Richard Lesh

Richard Lesh
Editor

Diane Mierkiewicz

Diane Mierkiewicz
Technical Editor

Foreward

The Georgia Center for the Study of Learning and Teaching Mathematics is a consortium of investigators dedicated to the improvement of the mathematical education of children. The consortium is currently organized into three Projects and nine Working Groups, listed below. Each Project and Working Group has general and specific goals. The general goals are common and constitute the goals of the Center as an organization. These goals were established as a result of a meeting of the Working Group Leaders in October of 1976. Specific goals are emerging for each Working Group. The goals of the Center are listed below.

Projects and Leaders	Conceptual Development of Mathematics	Problem Solving	Teaching Strategies
	Leslie P. Steffe	Larry Hatfield	Thomas J. Cooney
Working Groups and Leaders	Applications	Combinations of Heuristics	Analyzing Teacher Behavior
	Richard Lesh	J. Philip Smith	Thomas J. Cooney
	Models for Learning Mathematics	Task Variables	Protocols Development
	William Geeslin	Gerald Kulm	Kenneth Retzer
	Number and Measurement		Teaching Strategies
	Grayson Wheatley		John Dossey
	Rational Numbers		
	Thomas Kieren		
	Space and Geometry		
	Izzie Weinzweig		

(1) Establish and maintain a consortium of investigators dedicated to the improvement of mathematical education in the elementary and secondary schools through disciplined inquiry.

(2) Identify fundamental problems and issues within mathematics education.

(3) Conduct studies in mathematics education which contribute significantly to the organization of a particular field of investigation or to the resolution of fundamental problems and issues.

(4) Promote interinstitutional cooperation which most efficiently utilizes available human resources to address fundamental problems and issues in mathematics education.

(5) Promote communication among investigators in mathematics education and allied disciplines through publication of Center studies.

(6) Translate knowledge in mathematics education into a form useable by school practitioners.

This monograph represents a significant step toward realization of Goals 1-5 above. The investigators of the Space and Geometry Working Group are to be congratulated for their intensive efforts during the time the studies were in progress. A great deal of effort has been also expended by the faculty and students at Northwestern University in preparation of the monograph. It is only through such dedicated efforts that coordinated research can be conducted. The investigators of the Center view the utilization by school practitioners of knowledge generated through research efforts as both desirable and realistic. A next immediate step to be taken after the publication of a monograph such as this one is to generate interpretative reports of the theory and research. The Center Policy Board urges any individual interested in such interpretative reports to collaborate with the Center in preparation of the documents.

The Policy Board of the Center would like to thank ERIC/SMEAC of The Ohio State University for publishing this, and various other, monographs of the Center. This cooperation has greatly facilitated ongoing studies and has made the monographs published available to any interested individual. Other monographs and working papers are available and may be obtained by writing any member of the Policy Board.

The Policy Board

Leslie P. Steffe, Director   Thomas J. Cooney

Larry L. Hatfield   Jeremy Kilpatrick

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Introduction

In the Spring of 1975, under a grant from the National Science Foundation, the Georgia Center for the Study of Learning and Teaching Mathematics (GCSLTM) sponsored a series of five research workshops involving: (a) teaching strategies in mathematics, (b) number and measurement concepts, (c) space and geometry concepts, (d) models for learning mathematical concepts, and (e) problem solving. The goal of the GCSLTM workshops was to bring together people who had already been doing work in these areas with the hope of coordinating future research and curriculum development efforts. This monograph reports 13 studies that developed out of the space and geometry workshop.

Participants in the Georgia workshops believed that if progress is ever to be made on the complex issues that are important to mathematics educators, groups of people will have to develop better bases of communication so that individuals can profit by (and build upon) the work of others. Consequently, following each of the workshops, working groups were formed to continue the cooperative efforts initiated at the workshop. Specific objectives for each of these working groups, as well as project objectives for the GCSLTM, can be obtained by writing to the project's director, Professor Les Steffe.

Papers Presented at the Space and Geometry Workshop

To review a broad survey of past research, to develop a language for dealing with these problems, and to describe some of the most important directions for future research, the following papers were presented at the space and geometry workshop.

Mathematical Foundations of the Development of Spatial and
Geometrical Concepts..............................Edith Robinson

Piaget's Thinking about the Development of Space Concepts and
Geometry..Charles D. Smock

Breakthroughs in the Psychology of Learning and Teaching Geometry
....Izaak Wirszup

Recent Research on the Child's Conception of Space and Geometry
in Geneva: Research Work on Spatial Concepts at the International
Center for Genetic Epistemology.............Jacques Montangero

Needed Research on Space in the Context of the Geneva Group
..........................Jacques Montangero & Charles D. Smock

Cross-Cultural Research on Concepts of Space and Geometry
....Michael C. Mitchelmore

Transformation Geometry in Elementary School: Some Research Issues
....Richard Lesh

The proceedings of the space and geometry workshop were published in a book, Space and Geometry: Papers from a Research Workshop (Martin, 1976). In addition, several other Georgia workshops included papers pertaining to the acquisition of spatial and geometric concepts. For example, the "models" workshop (Osborne, 1976) included L. Martin's article, "The Erlanger Programm as a Model of the Child's Conception of Space," and the "number and measurement" workshop (Lesh, 1976) included several papers emphasizing the close links between the development of spatial/geometric concepts and the development of number/measurement concepts.

Most of the models and diagrams teachers use to introduce arithmetic and number concepts presuppose an understanding of certain spatial/geometric concepts. Consequently, misunderstandings about number concepts are often closely linked to misunderstandings about the models that are used to illustrate them. For this reason, one of the long range goals of the space and geometry working group has been to furnish information to help educators devise "better" instructional models for teaching basic arithmetic and number concepts.

Some General Issues Considered at the Space and Geometry Workshop

According to the papers presented at the space and geometry workshop, spatial/geometric concepts furnish an excellent context in which to study the following kinds of issues about the acquisition of mathematics concepts in general:

(1) What general principles can be found for anticipating the relative difficulty of mathematical ideas? For example, if a child is operational (in the Piagetian sense) with regard to one concept, what does this imply about the child's ability to learn other "related" ideas? Van Hiele, Freudenthal, and several Soviet psychologists, (e.g., Yakimanskaya and Sergeevich) have explicitly described the way they believe geometric concepts evolve in children; and Piaget's theory suggests it may be possible to analyze, order, and equate concepts (and models) on the basis of their underlying operational structures (Lesh, 1976). Yet, basic controversies and gaping holes occur in each of these theoretical descriptions; and the controversies strike at the heart of many of the most basic issues in developmental psychology. So, if it is possible to find techniques to anticipate the relative difficulty of geometric concepts, then similar techniques may be useful to organize the sequential presentation of arithmetic ideas--or instructional models leading to arithmetic concepts.

(2) Even within the category of "concrete materials," some materials are more concrete than others. For example, to teach "regrouping" concepts, each of the following materials can be used: (a) bundling sticks, (b) base ten arithmetic blocks, (c) base five arithmetic blocks, (d) a counting frame abacus, or (e) pennies, dimes, and dollars (play money). Yet, some of these materials are clearly more abstract than others; some draw on fewer intuitive notions; and some involve complex systems of relationships. Nonetheless, little has been done to investigate how the figurative content of a problem effects the difficulty of the task. Piaget has focused on the operational aspects of tasks and concepts and has deemphasized the figurative

aspects. However, the influence of figurative content on operational ability is important information for teachers who must devise models to illustrate mathematical concepts.

(3) Papers at the geometry workshop emphasized that, especially in geometry, children make many mathematical judgements using qualitatively different methods than those typically used by adults. Yet, the nature of these differences is not clearly understood. Consequently, research concerning the evolution of spatial concepts could help mathematics educators come to a better understanding of difficulties children experience concerning a wide range of mathematical concepts.

(4) One of the ingenious aspects of Piaget's theories is that he explicitly confronts the fact that ideas (as well as children) develop. That is: (a) a given idea can exist at many different levels of sophistication, (b) this evolution can be traced, and (c) the more primitive levels of a concept have seldom been investigated or accurately described. It is remarkable that so little is known about the nature of children's early conceptions of many mathematical ideas. This may be because the first mathematical judgements children learn to make are highly specialized, closely tied to specific content, and highly restricted; whereas the ideas mathematicians use as building blocks for their theories are those that are most general, those which can be used in the widest range of circumstances, and those which give rise to "nice" theories. If we try to use children's first concepts as building blocks for a theory--for example, as Grize (Beth & Piaget, 1966) has done using Piaget's grouping structures, we would find "messy" primitive concepts that do not give rise to neat, tidy, elegant theories. For this reason, mathematicians have not taken the trouble to formalize such awkward structures--especially as building blocks for a theory. In fact, it seems unlikely that mathematicians will ever take the trouble to describe most of the structures children use when they first come to master a given idea. Therefore, to develop effective instructional materials, it is critical to present ideas in a form that will be most understandable to children. So, it is important to formulate accurate descriptions of children's primitive conceptions of important mathematical ideas.

(5) Some of the best resources for describing the nature of children's early number concepts have come from Piagetian studies. Nonetheless, because Piagetian research has focused on the cognitive processes used by first-graders (i.e., concrete operational groupings) and by sixth-graders (i.e., INCR groups), children at intermediate levels of development have been neglected. Furthermore, because psychologists in general (and Piagetians in particular) have avoided mathematical ideas that are typically taught in elementary school, it is usually possible to make only relatively crude inferences about how children's mathematical thinking gradually changes from concrete operational concepts to formal operational concepts. It is time for mathematics educators to apply Piagetian techniques and theory to concepts that exist at intermediate levels of development--as well as at adult or preschool levels.

Activities of the Space and Geometry Working Group

Following the initial space and geometry workshop, a working group was formed to focus on projects concerning the issues described in the previous section. The space and geometry working group maintained communication through letters and a series of meetings--one at Northwestern University in January of 1976, another in Atlanta during the 1976 NCTM meeting, and other smaller meetings at the University of Georgia, Northwestern, or at mathematics education meetings like the 1976 International Congress of Mathematical Education. The purposes of these meetings were: (a) identifying important issues, (b) formulating basic (answerable) questions related to these issues, (c) planning projects to determine answers that are useful in a variety of situations, and (d) making plans to communicate the results and conclusions in a way that is meaningful to other mathematics educators.

As a result of the planning sessions described above, the 13 studies reported in the monograph were completed during the 1975-76 academic year.

The table of contents of this monograph suggests how the individual papers relate to the general issues described in the previous section. The articles by Lesh and Mierkiewicz, Lesh and Elwood, Fuson and Murray, Musick and Weinzweig all concern preoperational concepts of the development of primitive operational concepts. Weinzweig's article focuses on finding an appropriate mathematical description of some of these primitive mathematical concepts, and Musick's article focuses on the role of "activities" in early concept acquisition.

The articles by Thomas, Schultz, Kidder, and Perham concern the transitional phases between concrete and formal operations. In particular, these studies focus on elementary transformation geometry concepts and emphasize the role that figurative content can play in influencing the difficulty of Piaget-type tasks.

The articles by Fuson and Martin also concern transitional stages between concrete operations and formal operations. But, these two articles focus on affine transformations, similarities, or projections. The articles by Moyer and Johnson, and Dietz and Barnett, have to do with older subjects or formal operational concepts.

The above articles were critiqued and refereed by a panel including Art Coxford of the University of Michigan, Richard Lesh of Northwestern University, Alan Hoffer of the University of Oregon, Les Steffe of the University of Georgia, and Paul Trafton of the National College of Education. Members of the review panel also participated in the meetings where preliminary plans were formulated for most of the studies that were completed. The philosophy of the space and geometry working groups was that when experienced researchers work together for several days to isolate individual research projects, the chances increase that the projects that evolve will be more basic, more to the heart of the issues, and consequently more important. Furthermore, because many individuals will have input into the planning, and because many individuals will have an opportunity to coordinate their research efforts, more complex issues can be investigated. Finally, the space and geometry group believes that the optimal time to establish connections among individual research efforts is while project plans are in the formative stages of development--not a year or two after projects have been completed and reports finally appear in journals or at conferences.

Future Activities of the Space and Geometry Working Group

As an outgrowth of the 1975-76 activities of the space and geometry group, several meetings have been planned during the 1976-77 academic year. First, an "applied problem solving" conference was held at Northwestern University in December of 1976, and a "space and geometry" meeting and an "applied problem solving" meeting were held during the 1976 NCTM meeting in Cincinnati. In addition, the space and geometry group conducted a research reporting session at the Cincinnati NCTM meeting.

Although it may seem that "applications" and "problem solving" are fairly unrelated to space and geometry research, these topics arose as natural outgrowths of the space and geometry research that was conducted in 1975 and 1976. As the space and geometry working group focused on "Piagetian" operational analyses of various concepts and on analyses of the figurative contexts in which these concepts were used, it became clear that "having a concept" and "being able to use it in some contexts" often involve somewhat different processes. In addition to the figurative aspects of a problem situation which can alter its difficulty, the availability of certain general problem solving strategies, as well as certain affective characteristics also seem to relate to the solvability of a problem. Some of these issues will be investigated during the 1976-77 academic year.

Individuals who are interested in cooperating with the above research efforts should contact Richard Lesh at Northwestern University, Izzie Weinzweig at the University of Illinois, Circle Campus, or Les Steffe at the University of Georgia. From the beginning, the space and geometry group considered geometry to be a context in which to investigate general questions about concept acquisition. That is, research concerning the development of spatial concepts should not be an area which is only of interest to those who want to teach geometry.

Richard Lesh

Richard Lesh
Editor

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Perception, Imagery, and Conception in Geometry

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Northwestern University

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In instructional materials for teaching geometric concepts, and in research literature concerning the development of geometric concepts, there is a great deal of ambiguity about similarities and differences between perceptual and conceptual processes. For example, in recent years, it has been fashionable to refer to many children's "learning disabilities" as "perceptual problems." Yet, most of the supposed "perceptual problems" involve far more than perceptual processes; they also involve organizing and interpreting perceptual data and other conceptual processes. So, if a teacher wants to help youngsters learn a geometric concept, it is important to provide more than perceptual experience.

Some clear distinctions between perceptual and conceptual processes are needed. Generally, the word "perception" has been used to refer to situations where the senses gather information from the environment and transmit it to the brain. "Conception" has referred to situations where perceptual information is organized, interpreted, transformed, or otherwise acted upon. However. recent research in perception shows that these distinctions are naive. For example, even at the most primitive levels, perception theorists talk about conceptual-sounding processes like organizing and interpreting information, and even about hypothesis testing.

Some Basic Facts About Perception

> ...At every instant, the amount of information available to us is immeasurably great. Electromagnetic radiation of all wavelengths and combinations of wavelengths--radio waves, light waves, and X-rays--shower us all the time. Our environment is saturated with sounds, changes in air pressure and temperature, and changes in the chemical composition of the air. No real system, physical or biological, could possibly register and make use of all of this vast array of combinations of the physical conditions in the environment. The information that is actually registered or acted upon by a real organism is always a very small fraction of that which could be used by an imaginary perfect system. All organisms select part of the information in their environment to register or act upon, and the rest of that information is lost to them. (Cornsweet, 1970, p. 1)

> ...Clearly, an organism cannot process all the external signals that it receives. In man, in the visual system alone, there are more than a million channels...This figure far exceeds our capacity to deal with information, which is limited to about 25 "bits" of binary information per second. Thus, the information reaching the brain must be whittled down considerably in order to abstract the most relevant data. (Held & Richards, 1971, p. 4)

Perception theorists have shown that the brain cannot process more than a very limited number of items at one time. So, for the sake of efficiency, it is important that each item sent to it carries as much information as possible. There are several ways that perceptual processes perform this simplifying function. First, it selects only those inputs that are most relevant for the task at hand. That is, it weeds out information. A good example of the filtering mechanism is apparent in "cocktail party" situations when attention can be paid to one voice among many (Broadbent, 1962).

> ...A closely related phenomenon is that if we have stopped hearing some continuous sound and then the sound goes off, we notice the cessation. In fact, when a clock stops ticking, it seems as though we hear the last few ticks. (An old story illustrates an extreme case. At 11:46 in the evening, an electrical failure disabled Big Ben's chimes so that they could not sound. At 12:00 sharp, the man who lived next door leapt out of bed and shouted: "What was that?") (Cornsweet, 1970, p. 442)

Perceptual mechanisms not only weed out information, they also read in information by imposing meaning and order on sensori input. For example, in "reading" situations the meaningfulness of the input plays an important role in determining what is perceived. "Reading is not simply stringing symbols together. A better description is that reading involves generating hypotheses about the meaning of the pattern of symbols" (Kolers, 1972, p. 84).

Another example where meaning plays a role in perception occurs in the perception of speech when it is very difficult to "hear" two messages at once (e.g., Oh God say save can our you gracious see Queen) even though the necessary information is available. Simplifying, organizing, interpreting, and hypothesis testing are all natural perceptual processes. Consequently, "the senses and the nervous system do not convey an exact representation of the object exciting the senses, but only an abstraction" (Held & Richards, 1971, p. 3).

> ...Perception is the process of knowing objects and events in the world by means of the senses. Historically and popularly, we have thought of the process of perception as the transmission of a copy (picture) of an object to a sense organ and thence to the seat of consciousness in the brain. But from previous discussions of the reaction of the sensory systems, it is already clear that nothing like a replica is maintained in transmission, not even a replica of the space temporal pattern of energy distributed over such a receptor surface of the retina. (Held & Richards, 1971, p. 166)

Perception theorists have shown that information about the world does not enter the mind as raw data, but in an already highly organized and abstracted form.

> ...Visual perception involves "reading" from retinal images a host of characteristics of objects that are not represented directly by the images in the eyes. The image does not convey

> directly many important characteristics of objects: whether they are hard or soft, heavy or light, hot or cold.
>
> ...Perception seems to be a matter of looking up information that has been stored about objects and how they behave in various situations. The retinal image does little more than select the relevant stored data. This selection is rather like looking up entries in an encyclopedia: behavior is determined by the contents of the entry rather than by the stimulus that provoked the search. We can think of perception as being essentially the selection of the most appropriate stored hypothesis according to current sensory data.
>
> ...A look-up system of this kind has great advantages over a control system that responds simply to current input. If stored information is used, behavior can continue in the temporary absence of relevant information, or when there is inadequate information to provide precise control of behavior directly.
>
> ...If we consider the problems of storing information about objects, it soon becomes clear that it would be most uneconomical to store an independent model of each object for every distance and orientation it might occupy in surrounding space. It would be far more economical to store only typical characteristics of objects and to use current sensory information to adjust the selected model to fit the prevailing situation. (Gregory, 1968, p. 250)
>
> ...The naive observer believes that he correctly perceives the objects and events in the world and that is all that there is to perception. Doubts begin to arise only when properties that do not correspond to apparent reality are assigned to these objects and events...For example, is the measured size of an object directly proportional to its perceived size? Is the apparent speed of motion of a body exactly proportional to its actual velocity? That the answer to such questions is frequently "no" is apparent from our everyday experience: the remote figure of a man may appear to be no larger than an ant; after viewing a moving scene, an observer may perceive a truly stationary one to move in the opposite direction. (Held & Richards, 1971, p. 2)

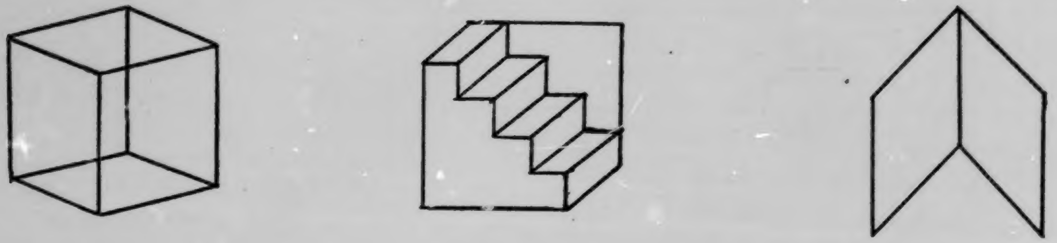
Perceiving is not just sensing, but rather the effect of sensori input on the representational system.

> ...the process of identification must involve some kind of matching between the visual input and a stored schema. If two schemata match the visual input about equally well, they compete for its perceptual interpretation; sometimes one of the objects is seen and sometimes the other. Therefore one reason ambiguity exists is that a single input can be matched to different schemata...It seems likely that the perceptual machinery is a teleological system that is "motivated" to represent the

> outside world as economically as possible, within the constraints of the input received and the limitations of its encoding capabilities. (Attneave, 1971, p. 66)

This type of perceptual competition is most clearly seen in "reversible figures," such as the illustrations in figure 1. The cube can reverse its depth, the staircase can go up to the left or across to the right, the folded card can appear like a "pup-tent" or like an open book; but none of these figure reversal interpretations can be seen simultaneously,

Figure 1



(a) cube (b) staircase (c) folded card

Perceptions are Constructed

Historically and popularly, men have thought of perception as the transmission of a copy (i.e., a picture) of an object. Because the human eye seems to act very much like a camera, photographs have been used as reasonable models of visual perceptions. However, according to modern perception theorists, there are many differences between the procedures that produce photographs and the mechanisms of perception. One difference is that a camera focuses by changing the distance from the lens to the film, whereas the eye changes the shape of the lens. Another obvious difference is that vision involves two eyes--not just one as in a camera lens. Yet the slightly differing information from the eyes is combined into a single perception.

It was not until the 17th century that the gross optics of image formation in the eye was clearly expressed. Until that time, many people believed that an inverted image on the retina was incompatible with seeing right side up. Today, most people comfort themselves with the thought that the brain compensates for inverted images. Actually, however, there is no problem--and so no compensation. Spatial patterns in the outside world are just patterns of nervous activity stimulated through the eyes (Wald, 1950, p. 32).

> ...If the analogy between eye and camera were valid, the thing one looked at would have to hold still like a photographer's model

> in order to be seen clearly. The opposite is true: far from obscuring the shapes and spatial relations of things, movement generally clarifies them...Eye movements are necessary because the area of clear vision available to the stationary eye is severely limited. To see this for oneself it is only necessary to fixate on a point in some unfamiliar picture or on an unread printed page. Only a small region around the fixation point will be clear. Most of the page is seen peripherally, which means that it is hazily visible at best. Only in the fovea, the small central part of the retina, are the receptor cells packed close enough together (and appropriately organized) to make a high degree of visual acuity possible. This is the reason one must turn one's eyes (or head) to look directly at objects in which one is particularly interested. (Neisser, 1968, p. 205)

By using a device somewhat like a contact lens with a small camera attached to it, scientists have been able to make an image stand still on the retina (Pritchard, 1961, p. 72). It was found that, when an image is stabilized on the retina by one means or another, it soon fades and disappears--later to regenerate in whole or in part.

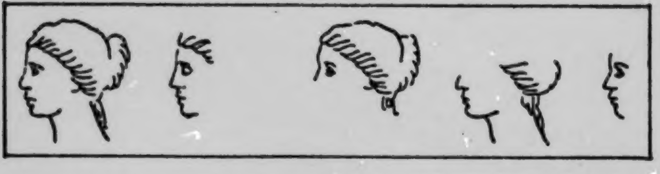
> ...In general we have found that the image of a simple figure, such as a single line, vanishes rapidly and then reappears as a complete image. A more complex target, such as the profile of a face or a pattern of curlicues, may similarly disappear and reappear as a whole; on the other hand, it may vanish in fragments, with one or more of its parts fading independently. (Pritchard, 1961, p. 72)

Pritchard (1961) has used illustrations like the ones in figure 2 to show how complex images fade and regenerate. To emphasize how learning plays a role in perception, Pritchard points out how the parts of a face tend to fade and regenerate in meaningful units while the parts of a meaningless configuration fade and regenerate rapidly in an (initially) unorganized and meaningless fashion. However, after a short period of time even the meaningless configurations tend to fade in organized units.

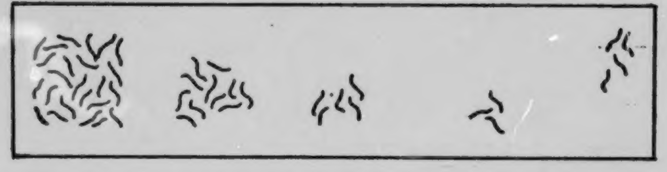
As Gestalt theory predicts, contiguity, similarity, field effects, and the dominance of "good" figures seems to effect stabilized images. However, Gestalt psychologists have maintained that objects are recognized as wholes, whereas more recent theories have proposed a more piecemeal process involving an assemblage of parts.

If perception involves an assemblage of parts, two important questions are: (a) What are the key features, or key relations, that are selected for identifying an object? (b) How are these features integrated and related to one another to form a complete perception of the object? Studies investigating eye movements during visual perception have produced results relevant to both of these questions.

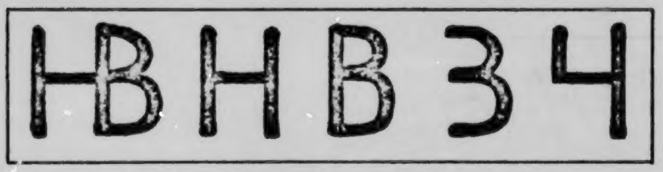
Figure 2



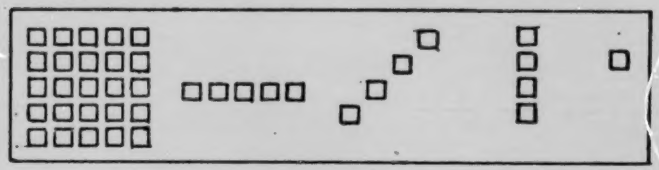
The parts of a profile drawing that stay visible are invariably specific features or groups of features, such as the front of the face or the top of the head.



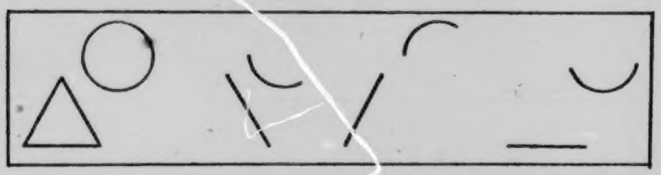
Meaningless curlicues first come and go in random sequence. But after a while small groups of curlicues organized in recognizable patterns start to behave as perceptual elements.



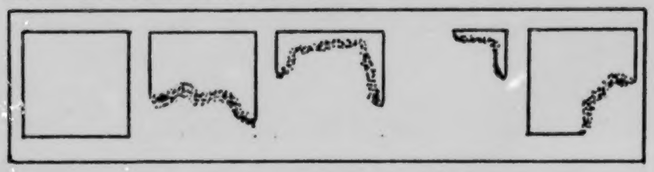
Monogram formed of the letters H and B also seems to illustrate the importance of elements that are meaningful because of past experience. When the monogram breaks up it is the recognizable letters and numbers within it that come successively into view.



Linear organization is emphasized by the fading of this target composed of rows of squares. The figure usually fades to leave one whole row visible; horizontal, diagonal, or vertical.



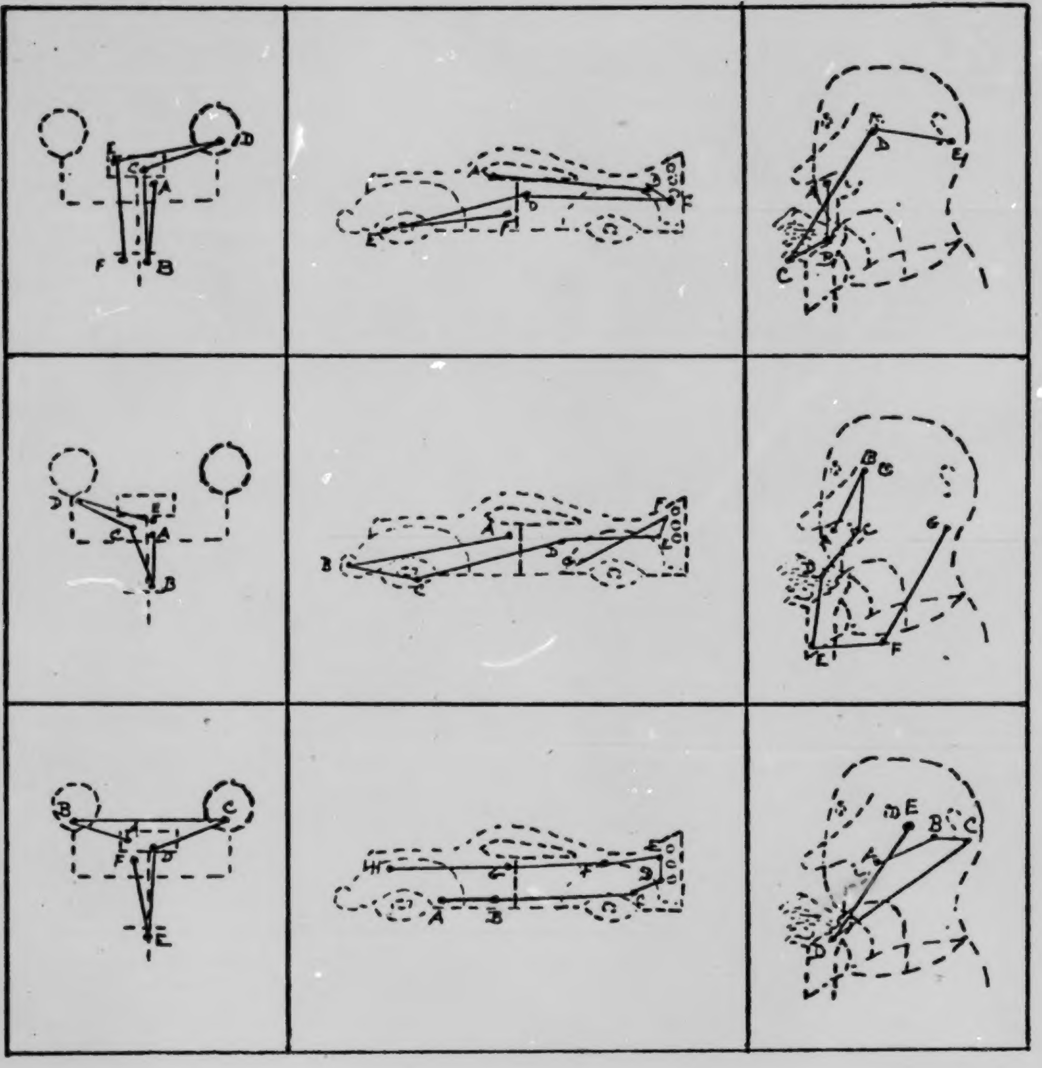
Circle and triangle may fade as units, leaving one or the other in view. When there is partial fading, a side of the triangle may remain in view along with a parallel segment of the circle, suggesting the "field effect" postulated in Gestalt visual theory.



Corners are the basic units when solid-tone figures are used. The fading starts in the center and the sharply defined corners disappear one by one. This target, like the others in the series, was presented to subjects both in white-on-black and black-on-white. (Pritchard, 1961, p. 75-77)

During normal viewing of stationary objects, the eyes alternate between fixations and rapid movements. For example, Noton and Stark (1971, p. 42) use the illustrations in figure 3 to show the scan paths for three different subjects and three pictures. By analyzing the scan paths for many different pictures and for many individuals, it has been shown that: (a) different people have different characteristic ways of viewing a given object; and (b) a given subject has quite different scan paths for different

Figure 3: Visual scanning paths



pictures. Nonetheless, certain regularities have been detected. For example, when subjects freely view simple pictures, their fixations tend to cluster around angles or points of greatest curvature.

> ...We conclude, then, that angles and other informative details are the features selected by the brain for remembering and recognizing an object. The next question concerns how these features are integrated by the brain into a whole--the internal representation--so that one sees the object as a whole, as an object rather than an unconnected sequence of features.
>
> ...It appears that fixation on any one feature, such as Nefertiti's eye, is usually followed by fixation on the same next feature, such as her mouth. The overall record seems to indicate a series of cycles; in each cycle the eyes visit the main features of the picture, following rather regular pathways from feature to feature... Essentially we propose that in the internal representation or memory of the picture the features are linked together in sequence by the memory of the eye movement required to look from one feature to the next.
>
> ...Our hypothesis states that as a subject views an object for the first time and becomes familiar with it, he scans it with his eyes and develops a scan path for it. During this time, he lays down the memory traces of the feature ring, which records both the sensory activity and the motor activity. When he subsequently encounters the same object again, he recognizes it by matching it with the feature ring, which is its internal representation in his memory. Matching consists in verifying the successive features and carrying out the intervening eye movements, as directed by the feature ring. (Noton & Stark, 1971, p. 38-39)

Sometimes (e.g., for small objects), no scan paths were detected. But, in these situations research suggests that a subject's attention moved around a picture even though his fixation remained steady near the center of the picture. Furthermore, the features of the objects that are selected for attention or fixation are again those which yield the most information (e.g., corners, etc.).

> ...Although seeing requires storage of information, this memory cannot be thought of as a sequence of superposed retinal images. Superposition would give rise only to a sort of smear in which all detail is lost. Nor can we assume that the perceiver keeps careful track of his eye movements and thus is able to set each new retinal image in just the right place in relation to the older stored ones. Such an alignment would require a much finer monitoring of eye motion than is actually available...It seems, therefore, that perceiving involves a memory that is not representational but schematic. During a series of fixations the perceiver synthesizes a model or schema of the scene before him, using

> information from each successive fixation to add detail or to extend the construction. This constructed whole is what guides his movements (including further eye movements in many cases) and it is what he describes when he is being introspective. In short, it is what he sees.
>
> ...Not only perception but also memory has often been explained in terms of an image theory. Having looked at the retinal picture, the perceiver supposedly files it away somehow, as one might put a photograph in an album. Later, if he is lucky, he can take it out again in the form of a "memory image" and look at it a second time. The widespread notion that some people have a "photographic memory" reflects this analogy in a particularly literal way, but in a weaker form it is usually applied even to ordinary remembering. The analogy suggests that the mechanism of visual memory is a natural extension of the mechanisms of vision. Although there is some truth to this proposition,...it is not because both perception and memory are copying processes. Rather it is because neither perception nor memory is a copying process. (Neisser, 1968, p. 204)

The Role of Perceptual Activity

Traditionally, perceptual development has been described from either a nativist or an empiricist point of view.

> ...On one side is the nativist, who believes that the infant has a wide range of innate visual capacities and predilections, which have evolved in animals over millions of years, and that these give a primitive order and meaning to the world from the "first look." On the other side is the extreme empiricist, who holds that the infant learns to see and to use what he sees only by trial and error or association, starting, as John Locke put it, with a mind like a blank slate. (Fantz, 1961, p. 66)

The empiricist point of view has been the most popular, particularly regarding the notions of space perception and constancy.

> ...The retinal image contains many cues to depth; for example, far-off objects are projected lower on the retina than nearby objects (which is why they appear higher to us). Supposedly, a baby learns that it must crawl or reach farther to get to such a higher image, and so comes to correlate relative height with relative distance. (Bower, 1966, p. 81)

Many empiricists believe that sensori-motor activities such as crawling and reaching are essential to perceptual development. However, the influences of sensori-motor activity on perception are even more evident in investigations of the adaptability of perceptual systems. Many of these studies have centered on the adaptation of visual perception to a radical transformation of the world induced through the use of prism goggles (e.g., the world can be made to look upside down, right and left can be reversed, or straight lines can appear curved) (Kohler, 1962; Held, 1965; Rock & Harris, 1967). Surprisingly, after a brief adaptation process, the visual system is able to acclimate to such distortions so that the goggle-wearer's perception appears normal. To explain these phenomena, Held (1965) notes:

> ...There is more to the mechanism of perceptual adaptation than a change in the way the sensory parts of the central nervous system process data from the eyes and ears. The muscles and motor parts of the nervous system are evidently involved in the adaptation too...(p. 84)

To support this hypothesis, Held compared the visual adaptation of subjects encouraged to actively explore their environment with those confined to passive movement. He found that "the degree of adaptation achieved by the subjects who had been involved in active movement was far greater than that of the passive group" (p. 88).

However, perceptual development involves more than just sensori-motor activity. After a series of studies with infants 1 to 15 weeks old, Fantz (1961) concluded that neither the nativist or empiricist views are accurate. Instead, "there appears to be a complex interplay of innate ability, maturation and learning in the molding of visual behavior..." (Fantz, 1961, p. 69-70). Hence,

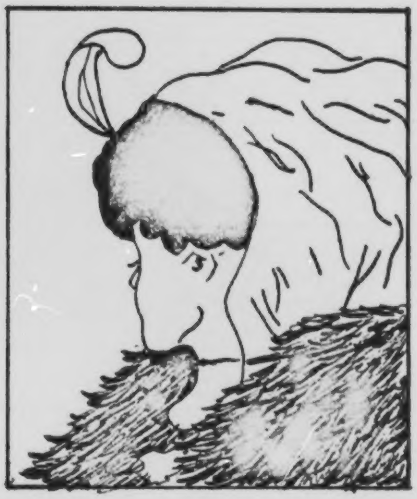
> ...It has long been assumed that perceptual development is a process of construction--that at birth infants receive through their senses fragmentary information that is elaborated and built on to produce the ordered perceptual world of the adult. The theory emerging from our studies and others not reported here is based on evidence that infants can in fact register most of the information an adult can register but can handle less of the information than adults can. Through maturation they presumably develop the requisite information-processing capacity. (Bowers, 1966, p. 92)

In other words, although an infant "receives" almost the same raw perceptual data as an adult, he has yet to organize it. So, the development of perceptual processes seems to involve the gradual coordination of information processing capabilities. This implies that sensori-motor activity will facilitate perceptual development when it aids in the organization of information.

Some Characteristics of Unorganized Perceptions

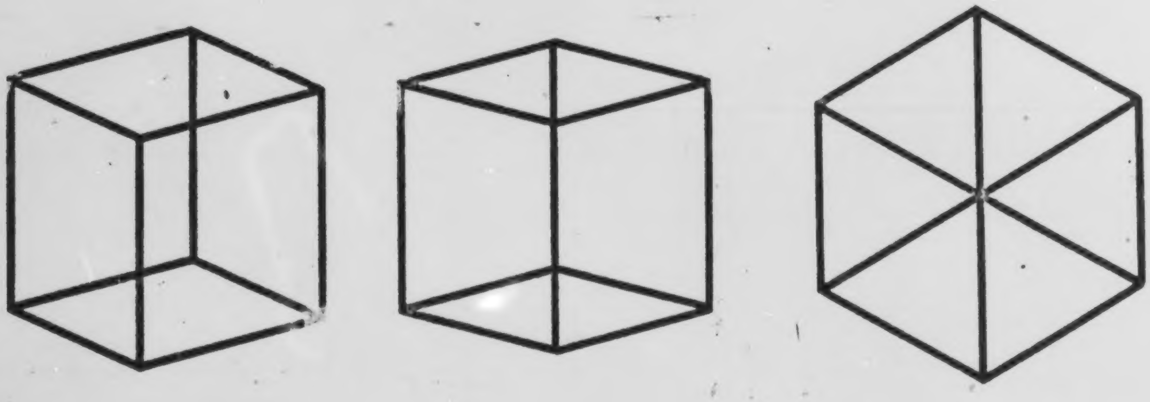
The preceding sections illustrated how perception involves conceptual sounding processes like organizing, interpreting, hypothesis testing, and memory. Consequently, just as in conception, the following phenomena typically occur when a person is forced to make a perceptual judgment in a situation where the relevant organizational schemes have not yet been coordinated:

Figure 4



(a) The subject may fail to "read out" all of the information that is in a given situation. For example, in a hidden picture puzzle like figure 4, one may at first see either an old woman or a young woman; then when both have been seen it is difficult to realize how one had ever been neglected. Similarly, in any complex picture some aspects will be neglected until the picture has been organized by the perceiver.

Figure 5



(a) (b) (c)

(b) The subject may "read in" too much meaning in a given situation, causing distortions or perceptual illusions. Or, a person may "read in" information that is actually helpful. For example, the illustration in figure 5c can be interpreted either as a flat hexagon or as a cube. And, the amount of information read in can be influenced by past experience. For instance, if a subject is shown figures 5a and 5b before figure 5c, the chances increase that figure 5c will be seen as a cube.

Another example of "reading in information" is illustrated in figure 6 where white squares and circles are "seen" even though none are actually present. Similarly, in figure 7 the subjectively imposed figures produce perceptual illusions just as though the imposed figures were actually present.

Figure 6

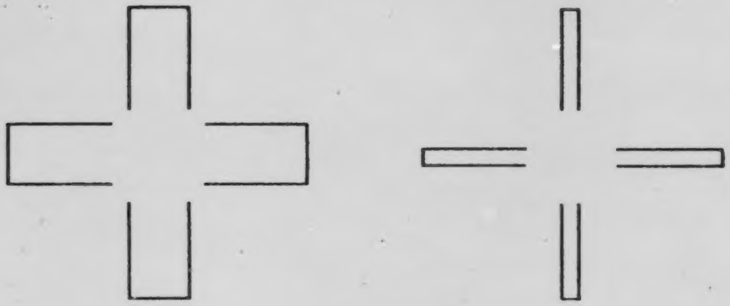
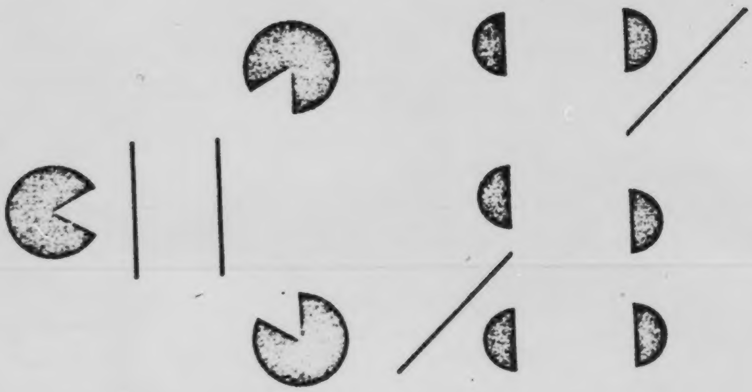


Figure 7



The total development of perception involves organization at two levels: (a) the organization of sensori information encoded, and (b) the organization of the perceptual activities used in gathering information. For example, in spite of the fact that "stabilized image" studies indicate that perceptual activity is involved even in the simplest situations, Bower (1966) has shown that infants 50 or 60 days old display shape constancy and can discriminate rectangles from trapezoids. Yet, in studies of haptic perception, Piaget (1967) has shown that children as old as four or five years had difficulty identifying various simple rectilinear shapes. Although this developmental "lag" can be attributed to a number of factors, such as the different senses used, one important variable was probably the amount of overt perceptual activity required in each task. In Bower's study, the stimuli were arranged so that a minimum amount of visual scanning (possibly one fixation) was sufficient to perceive all the relevant information. In Piaget's study, rather large shapes made it impossible for the children to perceive all the important data at once. Hence, to correctly identify the shapes, children had to actively explore the blocks and actively organize their exploration activities. Children who correctly identified the shapes used organized systems of data gathering activities.

It seems clear that the amount of perceptual activity to be coordinated effects the difficulty of perception tasks. For example, in this book, Fuson and Murray's "haptic perception" article shows that Piaget's tasks become much easier if the amount of organizational activity is reduced by using smaller figures which need not be explored so actively. Nonetheless, Fuson and Murray's study also indicates that children seem to go through the same stages of development as in Piaget's tasks--only at an earlier age. This suggests that even though the tasks were made easier by minimizing the importance of organized exploration activities, the same basic processes may be involved.

The total development of perception seems to require the organization and coordination of both the sensori information encoded and the perceptual activity gathering this information. However, while most perception theorists have focused on the former types of organizational processes, Piaget has emphasized the role of the latter. In fact, Piaget has devised tasks in which the degree of organized perceptual activity is so great that cognition structures are needed to coordinate the perceptual activities. Consequently, perception sometimes actually "trails" the organization of the intellectual concepts involved. Consequently, a child may fail to make a perceptual judgment from a situation even though he may understand the prerequisite intellectual concepts. For example, this developmental "lag" was shown in children's estimation of the relative size of objects placed at different distances from them (Piaget, 1969).

Some Basic Facts About Imagery

The preceding sections suggest that perceptual processes are very similar to the processes that are involved in constructing internal representations--or images. The word image is popularly used with a variety of different meanings: (a) the optical pattern on the retina, (b) the mental experience of "seeing" the object, and (c) the seemingly picture-like memory of certain objects or events. And, it has been quite easy to treat the inner image as a simple copy of the outer image. However, research clearly shows that this point of view is naive. Internal representations are not at all like the optical images on the back of the eye. Nonetheless, some interesting similarities have been found between type "b" and type "c" images. For instance, while examining the rapid eye movements that regularly accompany sleep, and by comparing these data with reported transcripts of the dreams, parallels were found between the eye movements of the dreamer and the content of the dream (Dement & Wolpert, 1958). Similarly, work on eidetic imagery* indicates that the Eidetiker scans the images with his eyes in much the same way that a real object would be scanned. After reviewing research of this type, Neisser (1968) concludes:

> ...First, seeing and imagining employ similar--perhaps the same--mechanisms. Second, images can be useful, even when they are not vivid or lifelike, even for people who do not have "good imagery." Third, mental images are constructs and not copies.
>
> ...The eye and brain do not act as a camera or a recording instrument. Neither in perceiving nor in remembering is there any enduring copy of the optical input.
>
> ...Visual memory differs from perception because it is based primarily on stored rather than on current information, but it involves the same kind of synthesis. (p. 208)

Piaget has been one of the foremost theorists emphasizing the constructed nature of perception, imagery, and conception. That is, perception, imagery, and conception require children gradually to coordinate and use progressively more elaborate systems of operations and relations. And, before these systems have been coordinated in a given situation, a subject can be expected to: (a) not "read out" some important information (i.e., centering), or (b) "read in" too much (or irrelevant) information (i.e., egocentrism). So, in case after case, children and adults are willing to distort what they see in order to fit what they understand.

Piaget (Piaget & Inhelder, 1971) explains that in the early days of experimental psychology, thought was considered to involve systems of associations between images--somewhat like a movie projector film. That is, images and associations were considered to be the two elements of thought. But, when the Wurzburg School demonstrated the existence of imageless thought, images were discarded and thought was reduced to associations. However, the early conception of images treated them as a residual trace of perception--which was in turn conceived as a photograph-like process. Images were not considered to be symbolic auxiliaries to thought, and they were not considered to require construction.

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* An eidetic image is an imaginative production that seems to be external to the viewer and to have a location in perceived space; it has a clarity comparable to that of genuinely perceived objects; it can be examined by the "Eidetiker," who may report details that he did not notice in the original presentation of the stimulus.

For Piaget, an image is a multifaceted symbol (Piaget & Inhelder, 1971). That is, to be a symbol, it must: (a) be able to be distinguished from the object it symbolizes, and (b) be able to be conjured up in the absence of the object it symbolizes. And, to be multifaceted, the symbol must be more than an arbitrary sign like "s" which might be assigned to a square for labeling purposes, but which conveys no further information about the object. For example, when a symbol like □ is used to represent a square, it not only serves a labeling function like the word "square" or the symbol "s," it also simultaneously conveys a great deal of other information--like the facts that squares have: (a) 4 corners--each equal, and each 90°; (b) 4 straight sides, with opposite sides parallel, etc. And, all of these bits of information come free with the symbol □ . Consequently, it qualifies as a multifaceted symbol--or an image.

Because of the "free information" aspect of images, it is easy to think of simple geometric problems where the solution can be found by "picturing a figure in your mind," and then operating on the picture to deduce new information--all without relying on formal properties of the objects. Probably it is this free information aspect of images that accounts for a large portion of the geometric intuition that mathematicians speak of so frequently.

Because of the multifaceted aspect of images, they bear some resemblance to the thing symbolized. Nonetheless, mental images are not so much attempts to produce exact representations of a thing "seen" as they are attempts by the subject to express what is "understood." For example, Piaget has devised a number of tasks to illustrate how some images develop rather late, and how children's representations often radically distort what they see in order to represent what they understand.

> ...spatial, geometric intuition is the only field in which imagined form and content are homogeneous...An image with logico-arithmetic content entails a conversion of non-spatio-temporal transformations into a necessarily spatial form. The spatial image, on the other hand, represents spatial content in forms that are likewise spatial... The image of a number or a class is not in itself a member or a class, but an image of a square is approximately square... (Piaget & Inhelder, 1971, p. 346)

However, even with regard to geometric concepts, there is never more than a partial isomorphism between the figural representation and the concept represented. So, images are often more like abstract symbols than like faded perceptions. For example, a picture of a cube must always distort some properties of a cube in order to represent others. In fact, two-dimensional representations of three-dimensional objects must always distort some properties or objects to emphasize others. For instance, each of the following drawings are equally "good" representations of a cube;

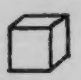
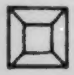
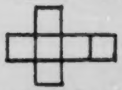
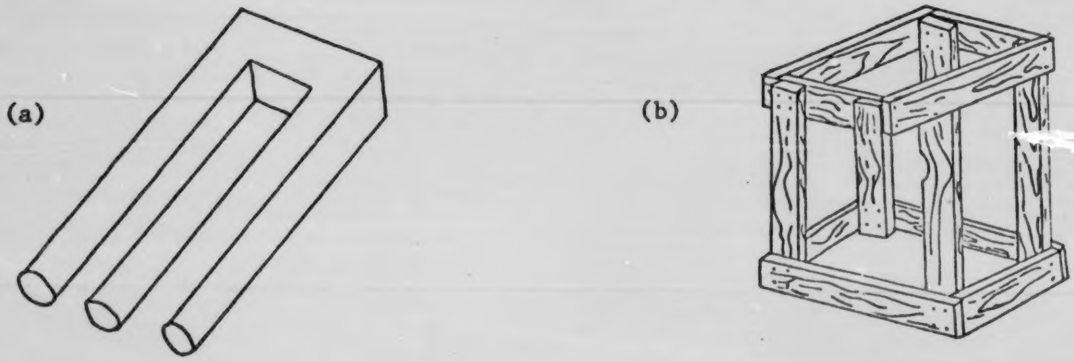
(a)  , (b)  , (c)  . Figures a and b distort the measures of the angles and the length of the sides, and figure c distorts connectedness to represent the squareness of the faces.

Figure a is probably more common than figures b and c, but it is no more accurate. In fact, for some purposes, figure c is best. For example, blueprints for houses (or directions for model airplanes) are more like figure c. Because any representation must distort some properties to emphasize others, judgements about "betterness" are always dependent upon the function the representation is to serve. But, many children have not yet learned to value the geometric properties their elders consider to be important. Consequently, children's judgements about the "goodness" of a representation may differ from those of adults. For example, in the case of perspective drawings of a cube (or selections from predrawn figures), children are usually more impressed by the squareness of the sides than they are by the measures of the angles, the connectedness of the sides, or the relative location of the sides. Consequently, their drawings and their internalized images distort properties they do not understand (and therefore do not consider important) in order to preserve properties they do understand (like the squareness of the faces). So, when they are asked for a perspective drawing, their drawings tend to resembe figure c (Montangero, 1976).

The examples in figure 8 illustrates situations in which adults typically distort what they see in order to fit what they understand. To verify this tendency, look carefully at figure 8a and then try to draw it when the figure is no longer visible.

Examples like figure 8b are quite common in the work of the artist M. C. Esher.

Figure 8



Adults become so accustomed to the usual "photograph-like" method of representing three-dimensional objects on two-dimensional surfaces that they forget that the method of using hazy backgrounds (and foregrounds) together with lines converging to a vanishing point (corresponding to the eye of the observer) was developed relatively late in the history of art. Early drawings tend to organize pictures using conceptual rather than optical relations. For example, early Egyptian drawings combined several different points of view within one scene (e.g., heads were painted in profile, with eyes painted in front view); and size was used to represent the importance of the object rather than the actual relative size of the objects (e.g., kings looked like giants, while servants, animals, and inanimate objects were dwarfed). In fact, early Egyptian "medley of viewpoints" drawings closely resemble children's drawings in many respects. Just like many children's drawings, the objects in some Egyptian pictures were drawn in a linear sequence, as though the figures were marching in a parade; and objects that were conceptually related were drawn close together rather than objects that were actually (i.e., spatially) close.

In a number of books and articles (Piaget & Inhelder, 1967, 1971; Piaget, Inhelder, & Szeminska, 1960; Ripple & Rockcastle, 1964), Piaget has distinguished between the figurative and operative aspects of thought. The figurative aspect deals with fixed states; while the operative aspect deals with transformations leading from one state to another, or with logical-mathematical operations and relations that are imposed on the elements of the fixed state. Consequently, the figurative aspect tends to be based on physical rather than logical-mathematical experience--and constitutes "empirical" reality. However, Piaget states that, "Sooner or later reality comes to be seen as consisting of systems of transformations beneath the appearance of things" (Piaget & Inhelder, 1971). And, this is why, "In adults, the figurative aspect is subordinated to the operative aspect--that is, we see each state as the result of a transformation of a previous state" (Ripple & Rockcastle, 1964, p. 21).

Piaget's conception of imagery is closely linked to perception in that both are closely tied to the figurative aspects of thought. Consequently, the development of imagery is in some respects parallel to the development of perceptual structures. However, imagery as well as perception is considered to involve far more than the photograph-like process that many others have considered it to be. For Piaget, images must be constructed, and the construction requires children to coordinate systems of operations and relations. This latter point is particularly important to emphasize because many modern psychologists (e.g., Paivio, 1971) have tended to neglect the constructed nature of images. By focusing on associative learning variables (like familiarity), these theorists have neglected the fact that the operational complexity of a figure is one of the most important variables determining the child's ability to use the image.

> ...With the image, as with perception, it is the sense data which "signifies," while the movements and their organization (in the form of comparative sensori-motor schemata) constitute the basis

> of the 'signified' relationships themselves. (Piaget & Inhelder, 1967, p. 42)
>
> ...images and sense data perform exactly the same function in geometrical intuition as in other thought processes. Namely, that of symbols or 'signifiers' as opposed to the relationships they 'signify.' (Piaget & Inhelder, 1967, p. 447)
>
> ...for mathematicians, intuition is far more than a system of perceptions or images. Rather is it the basic awareness of space, at a level not yet formalized. (Piaget & Inhelder, 1967, p. 448)
>
> ...The 'intuition' of space is not a 'reading' or apprehension of the properties of objects, but from the very beginning, an action performed on them. It is precisely because it enriches and develops physical reality instead of merely extracting from it a set of ready-made structures, that action is eventually able to transcend physical limitations and create operational schemata which can be formalized and made to function in a purely abstract, deductive fashion. (Piaget & Inhelder, 1967, p. 449)

Relationships Between Perception and Imagery, and Conception

Perceptual activities require children to use coordinated systems of operations and relations; concepts consist of the pure operational systems--devoid of their figurative content--that the images symbolized. Piaget has shown that the characteristic feature of mathematical concepts is that they inherently involve systems of operational or relational structures. In fact, from a Piagetian point of view, it is no exaggeration to say that mathematical concepts _are_ operational systems.

The above line of reasoning suggests that the first operational systems (i.e., the first mathematical concepts) children master should be closely related (perhaps isomorphic) to the operations and relations that are involved in children's early perceptual experience. In fact, Piaget regards the perceptual and operational systems as being partially isomorphic in the following ways:

> ...First, perceptual structures and operational structures in general relate in this way: the semi-reversibility of the one is partially isomorphic to the full reversibility in the other; there is one form of equilibrium for the first and another (better) form for the second, etc. Second, the perceptual constancies are clearly analogous to the representational conservations (e.g., Piaget, 1954b); in both cases there is a kind of genotypical invariance established in the face of phenotypical change. Third, quasi-perceptual "figural collections" seem to be the preoperational forebears of later logical classes (Piaget & Inhelder, 1959). And finally, there appear to be

> "preinferences" in perceptual activity which are not quite logical inferences but show partial isomorphisms to it (Piaget & Morf, 1958b). (Flavel, 1963, p. 234)

In other words, perceptual systems are "crude sketches or first drafts of better structural intellectual phenomena to come" (Flavel, 1963, p. 233). For example, the perceptual literature indicates that "angles" are among the features that perception selects to focus on in many objects; and, in his analysis of haptic perception, Piaget concluded:

> ...in the case of haptic perception; it is the analysis of angles which leads to the discovery of straight lines rather than the other way about. The earliest squares or triangles are simply circles distinguished by the addition of one or two angles. (Piaget & Inhelder, 1967, p. 70)

Similarly, after studying the evolution of several geometric concepts, Piaget concludes:

> ...There is no doubt that it is the analysis of the angle which marks the transition from topological relationships to the perception of euclidean ones. It is not the straight line itself which the child contrasts with round shapes, but rather that conjunction of straight lines which go to form an angle. (Piaget & Inhelder, 1967, p. 30)

Perhaps more research of this type would be helpful. In another article (Lesh, 1976), the first author has pointed out that the first operational systems children learn to use are messy structures that mathematicians have usually not bothered to formalize. Consequently, surprisingly little is known about the structure of the earliest conceptions children have of most mathematical concepts. So, if Piaget's assertion is valid that "The teaching of geometry could hardly fail to profit from keeping to the natural pattern of development of geometrical thought" (Piaget & Inhelder, 1971, p. vii), then perhaps it is time for mathematics educators to take a close look at the growing quantity of literature that is becoming available concerning the nature and development of perceptual systems. Current research seems to indicate that the first conceptual systems children use may be closely related to the systems of activities that are involved in earlier developing perceptual capabilities. However, it is not clear what a geometry curriculum would look like it it attempted to use concepts like "corners" and "boundedness" as primitive notions rather than traditional primitive concepts like points, lines, and sets.

In this book, Weinzweig's article represents one attempt to devise an acceptable mathematical description of some of children's early geometric concepts. However, more efforts of this type are clearly needed. If curricula are ever to be devised conforming to the "natural" development of geometric concepts, it will be necessary for mathematics educators to formulate better mathematical descriptions of this development.

Summary

It is interesting to notice that perception theory seems to be in the process of becoming more and more conceptual (involving constructs like "memory," "interpretation," and "hypothesis testing") at the same time that conception is often treated as though it involved little more than helping a child become aware of commonalities among a series of perceptions. Perhaps this point of view should be taken seriously--not because perception and conception are passive "photograph-like" processes, but because neither are passive processes. Perception theory furnishes a rapidly growing body of interesting information that could force educators to think about concept formation in ways that are less naive and more productive.

Perception, imagery, and conception all involve figurative and operative aspects. In perception, the operations are used to organize the figurative information; in imagery, the figurative aspect comes to be treated as a symbol to represent the relations and operations that were used to make perceptual judgements; and in conception, the ideas that evolve are the pure operative systems--devoid of figurative content. However, there is reason to believe that (a) the same basic types of operations and relations are involved in each of these areas, (b) the primitive relations that are used to make perceptual judgements are similar to the first relations children use to make conceptual judgements, and (c) when children have not yet learned to use these relations in organized systems, similar difficulties arise (i.e., centering, egocentrism, etc.). Therefore, perception literature can provide insight into the nature of children's earliest conceptual systems, which, in turn, could be the basis of curricula which more closely "fit" the capabilities of children's developing cognitive systems.

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Apparent Memory Improvement Over Time?!!

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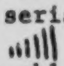
The task described in this article may seem rather unrelated to the kind of geometry exercises children encounter in elementary school. But, the issue involved is critical for educators who would like to use Piaget's theory to devise better instructional activities for children. The basic question concerns the way in which a child's operational ability limits his ability to read information out of figurative models (or drawings) that are used to illustrate basic mathematical concepts. Especially in geometry, to develop effective instructional activities, educators must come to a better understanding of the close relationship between the figurative and operative aspects of thinking.

Background Information Concerning the Study

In 1967, as the second in a series of Heinz Werner Lectures, Jean Piaget's address "Memory and Operations of Intelligence" (Piaget, 1968) briefly mentioned an "imagery" study in which children's memory of a configuration seemed to improve over time. Young children (3-8 years) were shown a seriated set of 10 sticks (9-15 cm), ordered from smallest to largest (). The children were asked to have a good look so they would be able to draw what they had seen at some later time. Then, one week later, without seeing the configuration again, they were asked to draw (or describe, or select from predrawn drawings) what they had seen. Similarly, after six months, without seeing the configuration, they were again asked to draw what they had seen. After the responses were classified according to type, the data showed a developmental trend in which younger children tended to represent what they understood rather than what they had seen. More surprisingly, however, in 74% of the cases, children's responses after six months were "better than" after only one week--the rest of the children showed no change; none had gotten worse. Piaget explains these phenomena in the following way:

> ...The interpretation which seems to be called for is the following. First of all, a memory-image is not simply the prolongation of the perception of the model. On the contrary, it seems to act in a symbolic manner so as to reflect the subject's assimilation "schemes", that is, the way in which he understood the model...Now in six months...this operational or preoperational scheme of assimilation evolves...Then the new scheme of the next level serves as the code for decoding the original memory. The final memory, then, is indeed a decoding, but it is the decoding of a code which has changed, which is better structured than it was before, and which gives rise to a new image which symbolizes the current state of the operational schema, and not what it was at the time when the encoding was done. (1968, p. 5)

Piaget reported that more than 20 variants on the above experiment were repeated using different materials and different types of instructions--some emphasizing language and some de-emphasizing language, some involving recognition memory, some involving reconstruction memory, and some involving evocation memory. Most of these variations yielded similar results. However, it was shown that "changes in the mnemonic code due to operational progress on the part of the subjects does not always lead to a better recollection six months later." Nonetheless, whether or not improvements occurred, the changes (or lack of changes) always seemed to parallel changes in a child's ability to use certain operational structures or schemes.

> ...Memory consists of two components. One of them is the figurative component, which is perceptual in the case of recognition, imitative in the case of reconstruction, and mental imagery in the case of the memory images necessary for evocation. The other is the operative component, which consists of action "schemes," or representational "schemes."...Everything that we have seen, in each of our experiments, shows the tight dependence of memory on the conservation and the development of "schemes." This is what explains the progress of memory over six months, where the "schemes" continue to develop, or the deterioration of memory where there is a conflict among two or more "schemes," or where the "schemes" are not adequate to support the memory-images. (1968,p.14-15)

According to Piaget, the apparent "memory" improvement resulted from an improvement in the operative aspect of thought, and was therefore really a phenomena of intelligence more than memory. Unfortunately, however, this rather simplistic interpretation conjures up the false impression that Piaget believes the figurative aspect of the child's thinking was "correct" (meaning a photograph-like copy of reality) from the beginning, and that the operative aspect simply had not yet developed sufficiently to decode this information.

Piaget and Inhelder's book Mental Imagery in the Child clearly emphasizes that: (a) The amount of figurative information a child is able to read out of a given situation is determined by the operational systems he is able to use. (b) Children will distort what they "see" in order to represent what they understand--which is in turn determined by the operational systems they are able to use. (c) Images are not photograph-like copies of reality, but are multifaceted symbols which must be constructed using systems of operations. So, the figurative and operative aspects of thought develop interdependently.

If all children, regardless of developmental level (i.e., operational ability), are able to accurately encode figurative information, and if their only problem is to increase their operative abilities so that these photograph-like images will take on new meaning, then Piaget's apparent "memory improvement" tasks are essentially trivial phenomena. It is well known that many kindergarten and preschool children are unable to copy a "staircase" arrangement of Cuisenaire rods even when a model is in plain view. So, it is not surprising that some of Piaget's children were unable to draw (build, or describe) the sticks when the only available model was in their memories. Furthermore, it is well known that four or five year old children typically change from "preoperational" to "operational" performance on seriation tasks like the one mentioned above. So, if encoding were able to give a perfect copy of the original figure, then it would be no surprise if (for some critical age level) decoding ability improved as a result of an increase in operational ability. But, this naive explanation would contradict one of the most basic tenets of Piaget's theory about image formation. That is, Piaget emphatically insists that operational ability is involved in the formation (i.e., the encoding as well as the decoding) of images--and that images are not photograph-like copies. Furthermore, Piaget (Piaget & Inhelder,1971) gives evidence to support the claim that the operations involved in the encoding of images are the same as those involved in decoding. So, if the results of the "memory improvement" tasks seem to argue that children are able to encode photograph-like copies in the absence of operational ability, then the results seem as detrimental to Piagetian theory as they are to more "associationist" theories.

The present study attempted to isolate some of the conditions under which apparent "memory improvement" can be expected to occur. In particular, it attempted to show that: (a) improvement is related to improved operational ability, and (b) encoding as well as decoding are affected by changes in operational ability.

Review of the Literature

Altemeyer et al. (1969) partitioned 100 kindergarten children into three treatment groups. Group A was shown an ordered sequence of dowel rods (9-16 cm in length|||||) and were asked to draw a picture of the sticks. Attention was drawn to the relative lengths of the sticks. The children were then asked to remember and draw what they had seen one week later--and again at six months. Group B was treated identically to group A except that at the one-week meeting questions were asked to draw attention to "distracting" characteristics (e.g., color, material) of the sticks. Group C was treated like group A except the children were shown an unorganized array of sticks (||||||).

A scoring manual was developed to categorize and evaluate responses. One of the difficulties with Piaget's procedure was that criteria for evaluating "betterness" were not explicit. It appeared that Piaget simply ranked drawings as correct and incorrect, ignoring levels of correctness.

Altemeyer's results showed virtually no difference between group A and group B. So, improvement did not seem to be affected by attention cues given by E. Among the 65 children in groups A and B, 28 improved, 21 remained the same, and 16 got worse. So, given the fact that one might expect children to remember very little from a brief experience after six months, there was some evidence for memory improvement. After one week, the most typical incorrect responses were: (a) one stick |, (b) more than one equal stick||||, (c) or different sized, unordered sticks |||||; and after six months, the most typical incorrect responses were: (a) ordered "arrowhead" arrangements ||||| or (b) ordered sequences with the wrong number of sticks ||||| .

Altemeyer reported that 65% of the "improvement" responses shifted at least two categories from non-patterned, non-ordered responses to relatively well ordered and patterned responses. Losses, on the other hand, were relatively minor--usually only one category below responses at one week; and most (i.e., 65%) of the losses were among children whose one-week responses were not only incorrect but also unordered and non-patterned. Children with correct drawings at one week usually also had correct drawings at six months.

Children do seem to be able to remember a great deal over a six-month period. Nonetheless, the apparent "memory improvements" may have little to do with memory. That is, in the study by Altemeyer et al., the apparent improvements could have occurred simply because of improved drawing ability. The hypothesis that memory may not be involved was strengthened by the fact that the responses for children in group C also tended to change from relatively unordered, unpatterned arrays to more ordered and patterned arrays. In fact, using the grading criteria that had been developed for the configuration for groups A and B, 41% "improved," 41% remained the same, and only 18% got worse--in spite of the fact that "betterness" is an inappropriate word to use when the initial array was actually unordered. Apparently, memory tends to become more patterned and ordered even when the initial configuration is not necessarily ordered or patterned. In fact, in Gestalt psychology, progressive ordering and patterning has been a well established fact for many years.

Altemeyer et al. also reported a brief follow-up study to investigate whether the group C "improvements" were simply a function of the fact that older children had a greater tendency to draw ordered and patterned arrays. Their data gave no evidence to support this hypothesis. They concluded that increased orderliness does seem to be related to memory and is not simply a function of drawing preferences of older children. Nonetheless, Liben (1975) replicated this latter portion of the Altemeyer study, and she concluded that "when less ambiguous directions are given (than in the Altemeyer study), there is indeed a developmental increase in children's tendency to spontaneously seriate stick drawings." So, the possibility of improvement due to drawing preferences remains an open question.

Dahlem (1968) conducted a study similar to the one by Altemeyer et al. except that the responses did not call for drawings. Instead, the children were given a set of sticks identical to the ones they had seen in the initial session, and their task was to remember and reconstruct the configuration after one week and again after six months. Again, a scoring system was devised to measure partially correct responses, and again the six month responses were better than the one week responses in 56% of the cases where the one week were not perfect. Only about 10% of the responses got worse between one week and six months. However, Dahlem did not use a control group like the one in the study by Altemeyer et al. So, perhaps the responses would have become "better" ordered even if the initial configuration had not been an ordered array. Or, perhaps older children would have a greater tendency to construct "staircase" configurations even if they had never seen the original configuration.

A second study by Dahlem (1969) investigated the possibility that apparent memory improvement resulted from the fact that children had one more practice session after six months than they had at one week. Dahlem also investigated the possibility that older children would have a greater tendency to make "staircase" arrangements even if they had never seen the original configuration. Treatment groups were given either 0, 1, 2, or 3 reproduction sessions during the first week. However, the data failed to support the hypothesis that memory drawing performance increased when more reproduction sessions were given. Furthermore, Dahlem (1969) showed that, among children who had not seen the original configuration, only 3.7% made staircase arrangements at the six month session. So, she concluded that improvement was related to memory and was not simply an artifact of structure that was built into the materials. Nonetheless, it is well known that kindergarten children have a tendency to spontaneously make staircase arrangements using Cuisenaire rods. So, because Cuisenaire rods are very similar to the dowel rods used in Dahlem's study, this conclusion still seems questionable. Furthermore, in Dahlem's second study, the evidence supporting memory improvement itself was rather weak. Although the number of perfect reproductions increased between one week and six months, when Dahlem attempted to go beyond simple comparisons of percent increases versus percent decreases, more sophisticated statistics failed to reach .05 levels of significance.

From her two studies on reconstitutive memory, Dahlem concluded that future studies should identify children having weak and strong structures. Presumably this suggestion was to allow the researcher to pinpoint the level of development where memory improvement would be most likely to occur. For example, Piaget and Inhelder (1967, p. 352) reported that after six months, 11 of 33 children aged 7-9 improved their memory drawings of the water level in an inclined bottle, whereas only 1 of 22 children aged 5-6 improved. According to Piaget, if children were at a level that was either too low or too high, memory improvement would be unlikely. Memory improvement would be most likely to occur for children who were at a transitional level of development during the six month period when the study was conducted.

Furth et al. (1974) conducted a reconstitutive memory study using four different types of line drawings (e.g., a tilted glass with liquid, a falling stick, an interrupted number sequence, and a house with a tilted chimney). Children were selected from grades K-4. So, grade level was in some sense an index of operative ability. Or, alternatively, the quality of initial drawings could be interpreted as a measure of operative ability. However, whichever measure of operative ability was used, no trends were noted showing differences in relative improvement or deterioration at different developmental levels. The general quality of drawings were age related across all tasks and all sessions, and memory drawing improved in 17% of the cases. But, memory drawings also deteriorated in 53-56% of the cases. The study concluded that the relative improvement/deterioration data resulted from an interaction between weakening figurative content and increased operative ability. Nonetheless, no improvement x age interaction was noted.

Other articles have been written dealing with reconstitutive memory phenomena (e.g., Murray & Bousell, 1970; Carey, 1971; Steinberg, 1974; Finkel & Crowley, 1973). Nonetheless, the basic issues have not been clarified. In some cases, memory drawing improvements do seem to occur, and in other cases improvement is questionable. Yet, little is known about the conditions in which improvement can be expected, and little is known about the role operative development plays in such increases.

Procedures

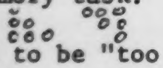
Selection and Categorization of Subjects

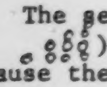
The study involved 169 kindergarten children from four schools in Evanston, Illinois' School District 65. Four Piagetian seriation tasks were used to partition the children into five operational ability levels in seriation. The lowest category included 26 children who were unable to copy a "staircase" of Cuisenaire rods. The second category included 47 children who were able to copy the Cuisenaire rods, but were unable to reconstruct an ordered "staircase" arrangement of 10 yellow dowel rods varying in length from 9 cm to 18 cm. The third category included 36 children who were able to reconstruct the dowel rod staircase, but were unable to correctly insert two "forgotten" intermediate rods (of lengths 12.5 cm and 15.5 cm) into a completed staircase. The fourth category included 26 children who were able to correctly insert intermediate rods into a completed staircase, but were unable to reconstruct a 4 x 4 matrix of dowel rods that varied in height along one dimension of the matrix and varied in width along the other dimension. The fifth category included 34 children who were able to correctly complete all four seriation tasks. This seriation task battery was developed in a study by Lesh (1975). A detailed description of the test items can be obtained by writing to the author. The test requires approximately 11 minutes to administer to individual children; and it has the property that in several previous studies, it has reliably sorted kindergarteners into five approximately equal sized groups according to seriation ability.

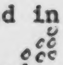
Assignment of Subjects to Treatment Groups

At each of the five seriation levels, approximately 2/3 of the children were randomly assigned to a group which would be given two memory tests--one after one week and another after six months. The remaining 1/3 of the children were only involved in the six month testing session. In this way, (a) the two scores for the first group could be compared to determine whether apparent memory improvement had occurred, and (b) the six month scores of the two groups could be compared to determine whether performance on the six month test had been influenced by experience from the one week test. Furthermore, both of these comparisons could be made either for the entire group (without regard to seriation ability) or within individual seriation levels. It was hypothesized that, if apparent memory improvement occurred at all, it would probably not be uniform across all seriation levels. This hypothesis was based on the notion that apparent memory improvement was presumably linked to improvements in operational ability and that children at the highest seriation level would already have developed all of the relevant operational abilities. To confirm this hypothesis, several pilot studies were conducted to develop a memory task where the following events would have an optimum probability of occurring: (a) maximum apparent memory improvement would occur for children at one of the middle seriation ability categories, (b) little apparent memory improvement would occur for the highest seriation ability level because all of the relevant operational abilities would already have been mastered, and (c) little apparent memory improvement would occur for the lowest seriation level because the children would be unable to adequately encode the relevant information in the first place.

Two Pilot Studies

The entire experiment reported in this article was repeated in three successive years--in each case with a slightly different configuration for the memory task. The initial study involved a double triangle array (i.e., ). But, the first experiment failed because the array seemed to be "too difficult." Nonetheless, the data did show that children are able to remember a great deal over a six month period; and some of the data suggested that a modified replication study might work if a simpler configuration were used.

The second study involved a triangle array of 10 poker chips (i.e., ). However, the second study was also a partial failure--again because the configuration seemed slightly too difficult. Slight improvement did seem to occur in some cases, but the data were "watered down" by a noticeably floor effect.

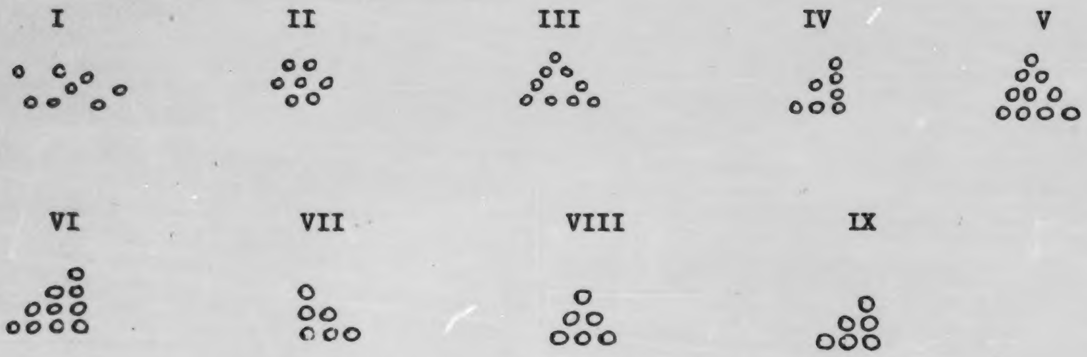
The third study, the one to be reported in this article, involved a triangular array of six poker chips (i.e., ). Poker chips were chosen so that, unlike materials like Cuisenaire rods and dowel rods, structure would not be "built into" the materials. That is, a child might spontaneously build a staircase with a set of Cuisenaire rods, but it is highly unlikely that the child would spontaneously make a six-poker-chip right triangle when given a stack of 25 poker chips.

The third study involved: (a) a construction task using concrete materials (i.e., poker chips), and (b) a task in which children were asked to select a correct configuration from arrays that had already been constructed. Drawing tasks were not used because pilot study experience indicated that young children's drawings were too varied, too unreliable, and too difficult to categorize. Furthermore, drawing tasks depend heavily on skills that are unrelated to cognitive development.

Selection of Preconstructed Configurations

Nine preconstructed arrays were selected on the basis of a pilot study involving 122 four year old preschoolers. Each child was given 25 poker chips and was asked to copy the triangle array of six poker chips shown in figure 1, IX. The contructions the children gave were then classified into the nine "types" shown in figure 1.

Figure 1

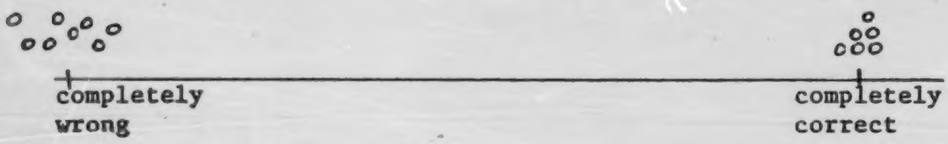


In figure 1, the type I response represents responses in which there was no perceivable pattern. Type II responses represent a class in which a well organized pattern was constructed which bore no relationship to the triangular model.

Evaluating the Quality of Preconstructed Configurations

In the final study, it would be important to be able to assign a "degree of correctness" rating for each of the constructions or selections children gave. Consequently, 66 "experts" were asked to evaluate the "degree of correctness" of the seven configuration types in figure 1, II-VIII. "Degree of correctness" was indicated by locating each configuration along a continuum like the one in figure 2. In figure 2, configuration I marked the location of "completely wrong" responses, and configuration IX marked the location of "completely correct" responses. All other construction types were ranked somewhere between these two points. (Note: The labels I, II, III, etc., were not assigned until after the experts had assigned "degree of correctness" rankings to each configuration type.)

Figure 2



The 66 experts were selected from: (a) faculty members and upper level doctoral students from Northwestern University's departments of psychology, educational psychology, mathematics education, and learning disabilities, and (b) from participants in the space and geometry research group from the University of Georgia Center for Learning and Teaching of Mathematics.

Table 1 shows the results of an analysis of variance on the among items and among individuals' evaluations of "degree of correctness." The low F among individuals reflects the high degree of agreement among the various experts. The high F among configurations reflects the facts that the experts generally considered the various configuration types to be significantly different from one another in degree of correctness. Newman-Keuls procedures showed that the only two configurations whose mean rankings were not significantly different ($p < .05$) were types VI and VII whose mean scores were nearly identical. Figure 3 shows the mean score results of the experts' rankings.

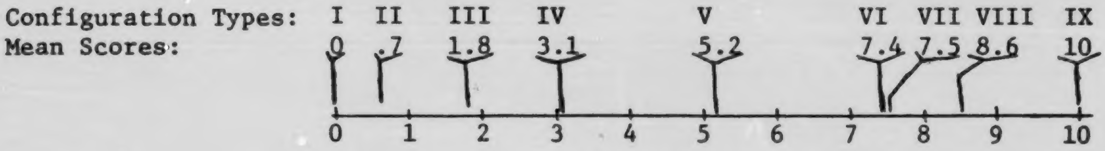
Table 1

Analysis of Variance for Seven Configuration Types
Evaluated by Sixty-six Experts

Source of Variation	df	MS	F
Among Configurations	6	6.811	2.902*
Among Individuals	65	2.108	.898
Residual	390	2.347	

* $p < .01$

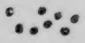
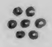
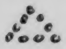
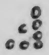
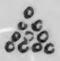
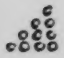
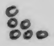
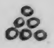
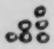
Figure 3. Mean "degree of correctness" scores for nine configuration types.



An analysis of the "degree of correctness" scores for the nine configurations revealed that the experts' evaluations generally seemed to correspond to the number of variables a given child represented in his construction or selection. Relevant variables seemed to include: (a) the overall triangular shape of the configuration, (b) the fact that the model configuration consisted of rows of objects, (c) the fact that the configuration also consisted of columns of objects, (d) the fact that the rows (or columns) increased in one step progressions, (e) the fact that the model consisted of six objects, (f) the "left-right" orientation of the figure, and (g) the fact that some organized pattern was attempted--even if partly incorrect. The results of this analysis are shown in table 2.

Table 2

Variables Considered in the Nine Construction Types

	I	II	III	IV	V	VI	VII	VIII	IX
									
Triangle Shape			✓	✓	✓	✓	✓	✓	✓
Rows					✓	✓	✓	✓	✓
Columns				✓		✓	✓		✓
One-Step Progression					✓	✓	✓	✓	✓
Six Objects							✓	✓	✓
Orientation				✓		✓			✓
Organization		✓	✓	✓	✓	✓	✓	✓	✓
Total Number of Variables Considered	0	1	2	4	4	6	6	5	7

The fact that the ordering of the objects did not correspond exactly to the number of variables probably resulted from the fact that the seven variables in table 2 were not considered to be of equal weight. Furthermore, some experts probably used criteria quite different from those listed in table 2. Nonetheless, interviews with the experts indicated that "number of variables considered" was an overall criteria used by most of the experts.

The Memory Task Sessions

During the third week of the school year, 169 kindergarten children were given the seriation task battery. The test took approximately 15 minutes per child to administer to individual children. Eight trained research assistants administered the tasks following a standardized procedure that is available from the author. The seriation test results were used to assign each child to one of five seriation levels. At each level, approximately 2/3 of the children were assigned to a group which would be given a memory test both at one week and at six months. The remaining children would be given only the six month memory test.

During the fourth week of the school year, each child was brought back into an individual session with the same research assistant who had given his/her seriation test. Each child was shown a triangle array of six poker chips (see configuration IX in figure 1), and was given 25 poker chips with the instructions, "Use your poker chips to make a design like this." Next, each child was asked to select a configuration "just like the model" from among nine preconstructed configurations like the ones shown in figure 1. Nearly all of the children were able to give correct responses to these two tasks. Only six children (all from the lowest seriation level) were unable to construct a correct copy on the first try; and even these six children were able to perform the task after some minimum guidance from the research assistant. Only five children (four of whom were from the lowest seriation level) were unable to select the preconstructed configuration that was like the model; and the primary problem seemed to be that these children were not clear about the nature of the task. Again, all five children were able to perform the task correctly on a second try.

The final step in the initial memory task session was to ask each child to: "Look at this design. Someday I will come back and ask you to make a design just like this one. So, look at it closely; and try to remember what it looks like."

One week after the initial memory task session (i.e., during the fifth week of the school year), two-thirds of the children in each seriation level were given the following two tasks (after some preliminary small talk): (a) Here are some counters. Make a design just like the one I showed you the last time I was here. (b) (After the child's design was recorded and removed from sight) Here are some designs some other children made. Which one is just like the design I showed you the last time I was here?

To end the second memory task session, the children were shown the correct configuration and were again told: "Look at this design. Someday I will come back and ask you to make a design just like this one. So, look at it closely; and try to remember what it looks like."

Approximately six months after the initial memory task session, all of the children were given the final session. The same two tasks were posed that were given in the second session.

Results

The results of the study are shown in table 3. For each of the 10 cells of the table, the rows of T_1 scores show the total number of subjects who made each response at the one-week testing session. The columns of T_2 scores include only the six-month scores of subjects who had also been tested during the one-week testing session. The T_3 columns include only the scores of subjects who were tested at six months but not at one week. The T columns show the sum of scores in the T_2 and T_3 columns; they show the total number of subjects who made each type of response at the six-month testing session.

Table 3

Seriation Level 5

Construction

	I	II	III	IV	V	VI/VII	VIII	IX	T2	T3	T
IX						1	1	6	10	5	15
VIII					1	1		3	5	2	7
VI/VII						1		2	3	1	4
V											
IV											
III											
II											
I											
T1					1	3	1	13	18	9	27

Recognition

	I	II	III	IV	V	VI/VII	VIII	IX	T2	T3	T
IX		1				1	1	10	13	5	18
VIII						1	1	1	3	2	5
VI/VII								2	2	1	3
V											
IV											
III											
II											
I											
T1		1				2	2	13	18	9	27

Seriation Level 4

Construction

	I	II	III	IV	V	VI/VII	VIII	IX	T2	T3	T
IX						1		4	5	1	6
VIII					2	2	2	1	7	1	8
VI/VII		1			1			1	3	2	5
V										1	1
IV											
III										1	1
II								1	1	1	2
I											
T1		1			3	3	2	7	16	7	23

Recognition

	I	II	III	IV	V	VI/VII	VIII	IX	T2	T3	T
IX						1	1	4	6	1	7
VIII			1			2	1	2	6	2	8
VI/VII		1					2	1	4	2	6
V											
IV											
III										1	1
II										1	1
I											
T1		1	1			3	4	7	16	7	23

Seriation Level 3

Construction

	I	II	III	IV	V	VI/VII	VIII	IX	T2	T3	T
IX						1	1	3	5	3	8
VIII			2	2	1	2	2	2	11	3	14
VI/VII				1	1			1	3	3	6
V					1	2			3	1	4
IV										1	1
III											
II											
I											
T1			2	3	3	5	3	6	22	12	34

Recognition

	I	II	III	IV	V	VI/VII	VIII	IX	T2	T3	T
IX						1	1	4	6	4	10
VIII			1			3	2	1	7	3	10
VI/VII		1	1	1			1	2	6	3	9
V											
IV						1			1	1	2
III						1			1	1	2
II			1						1		1
I											
T1		1	3	1		6	4	7	22	12	34

Seriation Level 2

Construction

	I	II	III	IV	V	VI/VII	VIII	IX	T2	T3	T
IX				1	1	2	2		6	3	9
VIII		1		2	1	2	1	2	9	4	13
VI/VII		1		2	1	2	2	1	9	2	11
V	2		1	1					4	2	6
IV										1	1
III			1				1		2	1	3
II					1				1	1	2
I										1	1
T1	2	2	2	6	4	6	6	3	31	15	46

Recognition

	I	II	III	IV	V	VI/VII	VIII	IX	T2	T3	T
IX			1			2	4	1	8	3	11
VIII		2	1	1		3	2	3	12	5	17
VI/VII		2				2	2	1	7	2	9
V											
IV			1			1			2	1	3
III		1	1						2	2	4
II										2	2
I											
T1		5	4	1		8	8	5	31	15	46

Seriation Level 1

Construction

	I	II	III	IV	V	VI/VII	VIII	IX	T2	T3	T
IX											
VIII				1		1		1	3		3
VI/VII				1	1		1		3	1	4
V				1	1	1			3	3	6
IV		1							1	1	2
III											
II	1	1	1	1			1		5	1	6
I	1								1	1	2
T1	2	2	1	4	2	2	2	1	16	7	23

Recognition

	I	II	III	IV	V	VI/VII	VIII	IX	T2	T3	T
IX						1	1		2		2
VIII			1	1			1	1	4	3	7
VI/VII		1		1		2		1	5	1	6
V											
IV						1			1		1
III		1							1	1	2
II										2	2
I		1	1				1		3		3
T1		3	2	2		4	3	2	16	7	23

In table 3, scores lying along the diagonal line of each cell represent subjects whose responses were the same at the one-week and six-month testing sessions. Similarly, scores in the upper half of each cell represent subjects whose scores were better at six months than they had been at one week, and scores in the lower half of each cell represent subjects whose responses were not as good at six months as they had been at one week.

Table 3 indicates that performance on both the construction and the recognition tasks were correlated with seriation level. Kendall correlations of score with seriation level were calculated for both one-week and six-month scores on both the construction and recognition tasks. All correlations were significant ($p < .001$).

The major hypotheses that performance on both the construction and recognition tasks would improve but that improvement would not be constant across seriation levels, and that performance would not differ across tasks were tested by a factorial (seriation level x type of task) analysis of covariance with repeated measures on both the variate (six-month scores) and the covariate (one-week scores).[*] Table 4 indicates that after the six-month scores were adjusted for differences in one-week scores, (a) seriation level had a significant effect on performance ($p < .001$); (b) the effect of the type of task approached significance ($p < .05$, but $p \nless .01$);[**] and (c) the interaction of type of task by seriation level was not significant.

These results indicate that a closer look at performance on each task within each seriation level is merited.[***] Furthermore, consideration of the results within each seriation level is of interest due to the qualitatively different degree of cognitive development reflected by each level. However, the "adjusted" mean scores for the six-month tasks were not used as measures of the performance of each seriation level group as is usually done in an analysis of covariance. This approach was abandoned for several reasons. First, because the one week scores were not normally distributed, the variance of these scores was not homogeneous, and the seriation level groups were not determined randomly, the underlying assumptions of the analysis of covariance were violated. Such violations can lead to biased results (Campbell & Boruch, 1975).

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[*] Preliminary analyses of performance on each task separately indicated that there was no interaction between one-week scores and seriation level for either task. The similarity of the regression coefficients for the construction and recognition tasks (.250 and .267 respectively) suggest that that the assumption of homogeneous regression coefficients for an analysis of covariance was met.

[**] Because a Kolmogorov-Smirnov test of normality indicated that the distribution of the one-week scores was non-normal ($p < .0001$), significance levels of .01 and .001 (rather than .05 and .01) were used in the analysis of covariance. Bartlett's Box F and F max tests indicated that the assumption of homogeneity of variance was not met ($p = .034$ and $p < .01$ respectively).

[***] Since the significance of the difference between tasks was "close" (i.e., $p < .05$ when the level of .01 had been selected), the individual seriation levels could differ across tasks; hence, performance within seriation level was investigated separately for each task.

Table 4

Analysis of Covariance

Source (adjusted)	SS	df	MS	F
Between subjects				
Seriation level	139.92	4	34.98	5.57**
Subjects within level	608.72	97	6.28	
Within subjects				
Type of task	15.32	1	15.32	6.06*
Type of task x seriation level	27.17	4	6.79	2.68
Type of task x subjects w/i level	245.12	97	2.53	

* $p < .05$

** $p < .001$

The additional problem of ceiling and floor effects further undermines the "truth" of any results obtained from an analysis of "adjusted" scores. Second, the six-month level of performance is itself not of interest. The critical question is the change between the one-week and six-month tests. Therefore, because the adjusted means do not reflect this change, gain scores (between one-week and six-month performance) were used as the basis for further analyses.

Table 5 shows the mean gain scores for each seriation level on each task. Testing each mean for significance ($\mu = 0$, $p < .05$) indicated that significant improvement was made on the construction task by subjects at seriation levels 2 and 3, and on the recognition task by subjects at seriation level 2. The mean gain scores over all seriation levels were also significant for each task ($p < .01$).

Table 5

	Seriation level	Mean score	df*	$\sigma_{\bar{x}}$	t
Construction tasks	1	.144	15	.7753	.186
	2	1.903	30	.6144	3.097***
	3	1.405	21	.6194	2.268**
	4	.463	15	.8530	.543
	5	-.044	17	.3585	-.123
	all	.959	102	.3009	3.1869***
Recognition tasks	1	.569	15	1.0236	.556
	2	1.487	30	.5746	2.588**
	3	.555	21	.6871	.808
	4	.694	15	.6268	1.107
	5	.439	17	.5911	.743
	all	.839	102	.3081	2.723***

** $p < .05$

*** $p < .01$

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* Since some of the groups were small, K-S test for normality was run on each group. The only significant deviations from normality were for the recognition tasks at level 2 ($p = .038$) and level 5 ($p = .026$).

The influence of the one-week task on the six-month scores was determined by a Kolmogorov-Smirnov 2-sample test of the six-month scores for each seriation level of each task and for each task across all seriation levels (see table 6). No significant difference was found for any seriation level or any task between the six-month scores of subjects who did the one-week task and those who did not.

Table 6

Kolmogorov - Smirnov 2-sample Test

Task	Seriation level	K - S z	2-tailed P*
Construction	1	.5123	.9555
	2	.5538	.9189
	3	.6333	.8174
	4	1.0245	.2446
	5	.2722	1.0000
	all	1.0218	.2474
Recognition	1	.4138	.9955
	2	.6496	.7927
	3	.1689	1.0000
	4	.7093	.6957
	5	.4082	.9963
	all	.8325	.4922

* H_o : The populations are the same.

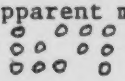
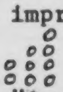
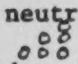
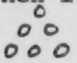
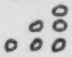
Conclusions

Overall, "apparent memory improvement" will occur under some circumstances; and the phenomenon is not a result of the fact that the one-week testing session provided a second learning situation. The group that was tested at one week and at six months did not differ significantly from the group that was only tested at six months.

In this study, four factors seem to influence apparent memory improvement: (a) forgetting, (b) the Gestalt-like tendency of memory to become "well ordered," (c) cognitive development over the six month period, and (d) the quality of initial encoding.

Forgetting: Not surprisingly, some forgetting did occur over a six month period. For example, many children who received perfect scores at the one-week session did not receive perfect scores at the six-month session; and, it is reasonable to assume that a similar forgetting factor was at least as strong (and perhaps stronger) for children who did not receive perfect scores at one week. Children receiving perfect one-week scores tended to be in the higher seriation levels.

The Tendency of Memory to Become "Well Ordered": Although no statistical tests were able to show significance, table 3 does suggest that children tended to "prefer" symmetric, well ordered figures (like II, III, V, or VIII) rather than nonsymmetrical figures (like I and IV--or, to a lesser extent, VI and VII)--and that this trend was more obvious at six months than at one week.

Because the entire study was conducted three times with different figures each time, the author gained some intuition about what types of figures would produce "apparent memory improvement." The figures in the first two studies (i.e.,  and ) were "too difficult" and so general poor performance modulated any "improvement" that may have occurred. Nonetheless, the general trends were consistent with the results reported here except that, for the more complex figures, there was an even greater tendency to select symmetric/well ordered figures. In fact, in order to neutralize this Gestalt-like effect, a partly nonsymmetric figure (i.e., ) was used in the final study. If the goal had simply been to produce high "apparent memory improvement" scores rather than to isolate variables that produced the improvement, then it would have been wise to have used a highly symmetric figure (i.e.,  rather than ). However, in the present study, Gestalt-like "good form" seemed to be a relatively neutral factor regarding "apparent memory improvement." For children who had given correct responses at the one-week session, "good form" reorganizing acted as a negative influence, whereas for children whose one-week responses had been "poor," the "good form" factor sometimes may have acted to help produce better responses.

Cognitive Development Over Six Months: The results clearly show that scores on both the one-week and six-month tests were closely correlated with seriation ability. In fact, at the lowest seriation level, several children were unable to copy the configuration even when it was in plain view. (At higher seriation levels, none of the children had difficulty copying the array.) Furthermore, the amount of improvement was also related to seriation ability. That is, the most improvement occurred at seriation levels 2 and 3, with very little improvement at levels 4 and 5, and a slight regression in performance at level 1.

The relative lack of improvement for seriation levels 4 and 5 presumably resulted from the facts that (a) children at levels 4 and 5 had already acquired most of the relevant operational abilities, (b) children at levels 4 and 5 had already scored quite well on the one-week test, so their six-month scores could only get worse. However, similar factors did not seem to produce improvement for children at the lowest seriation level--in spite of the fact that children at seriation level 1 had the most to gain in terms of improvements in operational ability and the least to lose from ceiling effects on gain scores.

The Quality of Initial Encoding: Operational ability (or simply organizing ability) could be involved in at least three phases of the learning/remembering process: (a) at the encoding phase, (b) reorganizing the information for storage, and/or retrieval, and (c) at the decoding phase.

The fact that similar phenomena occurred for construction and recognition tasks (with recognition tasks being slightly easier) suggests that "apparent memory improvement" is not simply the result of improved decoding ability (e.g., improved drawing or construction ability). So, whatever produces "apparent memory improvement," it is related to improvements in operational ability and it functions even when decoding ability (i.e., drawing or constructing) is minimized--as in the recognition task. This suggests that the operations that are involved at the encoding and reorganizing phases may be similar (or identical) to operations that are involved at the decoding phase. However, the operations that are involved function in systems that must be coordinated--and the decoding phase requires a higher degree of coordination than the encoding phase. Encoding would be analogous to painting a picture "by the numbers" when an outline is already given. Decoding would be analogous to painting the same picture, using the same movements, where the only outline is "in your head." Presumably, this is why children at the lowest seriation level did not improve over time like children at levels 2 and 3. At levels 2 and 3, the children's operational ability was sufficient to encode the relevant information correctly, but was insufficient to cope with the increased coordination required at the decoding phase. Consequently, after six months, when operational abilities had increased, children at levels 2 and 3 were able to decode the information they had encoded six months earlier. However, for children at level 1, the relevant information had never been encoded properly--so improvement could not occur.

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The Haptic-Visual Perception, Construction, and Drawing of Geometric Shapes by Children Aged Two to Five: A Piagetian Extension

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Piaget and Inhelder (1967) reported that children were able to make a haptic-visual discrimination of uncomplicated Euclidean shapes at about age five. However, infants can learn to make visual-visual discriminations of such shapes (Volkelt, 1930), and American upper-middle-class children commonly play with and learn to master the visual matching of shape toys involving up to twelve shapes by the age of two or two and one-half. The present study was designed to examine some of the variables which might be responsible for this large gap between visual-visual and haptic-visual discrimination.*

The size of the shape used seems to be an important variable. Piaget and Inhelder used relatively large shapes (their circle had a diameter of 11.5 cm.). There are two problems in using large shapes with young children. The first is that the child will miss some of the distinctive features of the shape in his haptic exploration. Piaget and Inhelder reported that their three and four year old children made very unsystematic searches; Zinchenko and Ruzskaya (1962) also found that three and four year olds make short haphazard haptic searches of objects. The second possible difficulty with large shapes is that the child will find all of the distinctive features in his haptic search, but will be unable to coordinate them into a single mental image which can be used to identify the shape visually. The shapes used in the present study were small enough to fit entirely into the hand of a two year old child. With such a small shape, the child would be unlikely to miss any distinctive features, and any successive focusing necessary to coordinate different features into a single image could be done quickly because he would not have to move his hand from feature to feature.

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* In this paper the following terminology will be used:

Haptic-visual discrimination refers to the presentation of objects for touch or manual exploration with their identification made from a group of visually-presented test objects.

Visual-visual discrimination refers to the visual presentation of objects with their identification made from a group of visually-presented test objects.

Visual-haptic discrimination refers to the visual presentation of objects with their identification made from touching or feeling unseen test objects (that is, haptic exploration).

Haptic-haptic discrimination refers to the presentation of unseen objects for touching with the identification of those objects made by haptic exploration.

Another possible variable influencing the seemingly late success of Piaget and Inhelder's subjects is their socio-economic background, which was not specified. For the present study, a sample of middle-class children in day care settings and upper-middle-class children in half-day nursery school settings was obtained. The latter sample of children might be expected to perform better both because their home environments might be richer and because they had more opportunity for individual interactions with a primary care-taker (their mother).

Piaget and Inhelder checked the development of children's haptic identification in three ways: by selection from drawings, by construction of the shape with sticks, and by drawing of the shape. They reported that success using the selection from drawings occurred slightly earlier than haptic success, but that the latter two methods gave the same results. However, a pilot study conducted by the current authors indicated that constructing shapes with sticks was considerably easier than drawing these shapes. Therefore, this study was also designed to assess the development of the abilities (1) to choose from a visual assortment a shape being examined haptically, (2) to construct a visually-presented shape, and (3) to draw a visually-presented shape.

Piaget and Inhelder investigated the child's ability to identify both familiar objects and geometric shapes using haptic exploration. Four of the geometric shapes used by them were chosen for this study: a circle, square, equilateral triangle, and diamond. The chronological order of success found by Piaget and Inhelder for these shapes was circle first, followed later by the square and triangle, and considerably later by the diamond.

Review of the Literature

Piaget and Inhelder (1967) reported the following stages in the haptic perception of form:

Stage 0 (< 2:6): Experimentation with hidden objects is not possible.
Stage 1 (3:0-4:4): Familiar objects are identified. Shapes requiring identification of topological relations (open, closed; number of holes, etc.) are identified. Shapes requiring identification of Euclidean relations are not identified.
Stage 2 (4:4-6:0): Rectilinear shapes differentiated from curvilinear shapes. Later in the stage simple shapes possessing Euclidean relations are discriminated (square, rectangle, triangle).
Stage 3 (> 6:0): Complex forms such as the swastika, cross, Cross of Lorraine, and star are recognized and drawn.

Several replications of the Piaget and Inhelder haptic study have been made (Fisher, 1965; Hoop, 1971; Laurendeau & Pinard, 1970; Lovell, 1959; Page, 1959; Peel, 1959). These studies were in large part concerned with the Piagetian assertion that topological relations are used by children before Euclidean ones, and that this development is reflected in their haptic identification of shapes. One interested in the controversy concerning the development of topological and Euclidean notions in children might wish to examine both the above studies, particularly Laurendeau and Pinard (1970) and Martin (1975) which analyzes Piaget's choice of figures from a mathematical perspective. For our purposes, the impact of these studies was in their support of Piaget's results. The socio-economic characteristics of the samples and the sizes of the shapes were not specified adequately in all of the studies, but higher socio-economic status and somewhat smaller shapes seemed to contribute to a younger age of success (Fisher, 1965; Milner & Bryant, 1970).

A considerable amount of research has been done in cross-modal discrimination of shapes in primates and in children. The wide assortment of variables involved in these studies and the inadequate specification of task variables makes comparison or succinct summary difficult.* However, these developmental studies do seem to indicate that visual-visual discrimination occurs first, followed by haptic-visual and visual-haptic discrimination, with haptic-haptic discrimination being the most difficult. Variables which seem to be associated with earlier success on cross-modal tasks are smaller shapes (sizes ranged from about four cm. to twelve cm.), simple shapes (circle, sphere), comparison test shapes that differ greatly from the initial shape, simultaneous presentation of the test objects, a small number of test objects, and simultaneous presentation of the initial and the comparison objects. Rose, Bland, and Bridger (1972) reported haptic-visual success by three and one-half and four and one-half year olds on simple discriminations between two quite dissimilar objects; but most studies reported a high level of success only for children five years of age or older. It should be noted that many of these studies involved time delays of some type.

The construction of geometric shapes with sticks has been little researched. Piaget and Inhelder reported that for only one out of 30 children examined were the matchstick constructions superior to the drawings (1967, p. 78). They found that errors in individual children's drawings paralleled similar errors in construction. The criteria used in judging either an adequate construction or a drawing were not specified, however, so it is difficult to tell just what was measured. Lovell (1959) asked his subjects to construct figures with matchsticks. He found the square easiest to construct, followed by the triangle and then the diamond. In a comparison of mean ages at which drawing and construction success occurred, he found that construction success occurred approximately six months before drawing success.

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* For a more complete discussion of this literature, see Fuson & Love-Kunesh, 1977, or write the first author.

Piaget and Inhelder (1967) reported the following stages in the development of the child's ability to draw shapes:

Stage 0 (Until 2:11): Drawings display no aim or purpose; they are mere scribbles.
Stage 1 (2:11-4:00): Topological relations such as open/closed are used. The circle, square and triangle are undifferentiated closed curves.
Stage 2 (4:0-6:6): This stage displays a progressive differentiation of objects based on Euclidean relations. At first curvilinear forms are drawn differently from rectilinear forms. Later in this stage relations such as length and number of sides are used, so that the circle is separated from the ellipse as is the square from the triangle. At the end of this stage, the rhombus (diamond) is drawn correctly.
Stage 3 (Above 6:6): There is little difficulty in drawing even the most complex forms, such as the swastika.

Piaget and Inhelder did not specify the criteria used for classifying the drawings, nor did they provide further details of the ages at which children draw specific shapes.

The research on children's drawing that is relevant to this study is of three kinds: replications of Piaget's drawing tasks, cognitive development studies, and tests used to assess a child's level of development. The Piagetian replications are largely concerned with the relative ease of drawing topological and Euclidean relations, but they do contain some data relating to the circle, square, triangle, and diamond. Lovell (1959) found that out of 21 shapes and figures drawn by children between the ages of three and five, the circle was second easiest to draw, the square sixth, the triangle sixteenth, and the diamond twenty-first. The triangle and diamond were more difficult to draw than figures containing two shapes, such as a triangle inscribed in a circle and a tiny triangle contained in the center of a circle, though this result may have been partially created by Lovell's scoring criteria. Peel (1959) reported that children's drawings of Piaget's 21 shapes fit a Guttman scale -- that is, that older children drew better than did younger children. Peel did not report the order of difficulty of the particular shapes.

Among other tasks, Birch and Lefford (1967) had subjects draw an isosceles triangle pointing up and pointing down, a diamond, and a square balancing on one vertex (a square diamond). Four conditions of drawing were used -- freehand drawing, tracing, copying on a dot grid, and copying on a square grid. The freehand condition was the most difficult at all ages. Birch and Lefford reported improvements in the freehand drawing of the triangles from the age of five to an asymptote appearing at age nine. Improvements in the freehand drawings of the two diamonds continued until age 11. For all forms, the greatest amount of improvement occurred between the ages of five and seven.

Olson (1970) conducted a series of studies on the child's acquisition of the concept of diagonality. He found that the diagonal is a very difficult concept and that success reproducing it occurs two to three years later than for vertical or horizontal line segments. This research indicates that one reason the triangle and diamond are more difficult to draw than the square is due to their inclusion of diagonal lines.

Developmental assessment instruments which include drawing of the circle, square, triangle, or diamond include the Stanford-Binet Intelligence Scale (1960), the Riley Pre-School Development Screening Inventory (1959), and the Test of Visual-Motor Integration (1967). In general, it is difficult to relate the results of these tests to those of Piaget because the criteria they use are inappropriate or ambiguous. For example, all of the tests norm the drawing of the circle at three years, but achievement is defined as predominantly rotary or circular lines which may or may not be closed. One of the key features Piaget required for successful circle-drawing was closure.

The tests vary considerably in their criteria for successful drawing of squares, and consequently norm success at different ages. The Riley defines a square as a mostly closed figure which includes at least three corners which are approximately at right angles; this "square" is drawn by four years of age. The VMI defines a square as a figure containing four distinct sides with corners not necessarily angular. This kind of square is drawn by females at 4.3 years and by males at 4.6 years. The Stanford-Binet finds that a square is drawn at five years of age. This test defines a square as four unbroken and perhaps slightly bowed lines, the heighth of which are no more than one and one-half times the width. These lines meet in unrounded corners.

The triangle is normed at six years of age by the Riley. The triangle is defined as a figure containing at least two angles drawn at approximately 60°, with unrounded lines that meet at distinct points. The VMI norms a triangle at 5.3 years. This triangle simply contains three distinct lines in which one corner is drawn above the other two. The Stanford-Binet does not include a triangle drawing.

Both the Riley and Stanford-Binet norm the diamond at seven years. The Riley defines a diamond as three distinct lines with approximate angles in which at least two corners meet at points. The figure is closed and upright. The Stanford-Binet defines a diamond as a figure containing four angles drawn opposite to each other. The figure may not be square- or kite-shaped. The VMI norms the diamond at 8.1 years. Its criteria are the same as the Stanford-Binet with the addition that the acute angles may not be more than 60°. The VMI states that the less mature diamond drawings include acute angles which are too large.

Although these tests vary in their scoring criteria and age norms for different shapes, they all agree that the order of difficulty for the four shapes is circle, square, triangle, and diamond.

A provocative discussion of the lengthy time lag between visual discrimination and shape drawing ability is contained in two papers by Eleanor Maccoby. In the first paper, Maccoby and Bee (1965) asserted that visual discrimination does not require the simultaneous recognition of all shape attributes, whereas shape reproduction via drawing does. This stance was modified in a second paper (Maccoby, 1968) which proposed that the form perception of infants and young children is holistic in that their discriminations are made "on the basis of the whole shape quality rather than on the basis of selected attributes" (1968, p. 165). This holistic perception is adequate for visual discrimination but not for copying, primarily because copying required the sequential execution of parts of a figure. Maccoby reported an experiment in which three, four, and five year old children were asked to match parts of a square, triangle, and diamond, and to draw these shapes. Children who were unable to find matches for horizontal or vertical lines produced poor drawings of all three shapes. Children who made errors in matching diagonal lines produced significantly poorer triangles and diamonds. In addition, training in discriminating right- and left-slanting diagonals from other lines led to improved performance on the drawings of the triangle and the diamond, but not the square (because it has no diagonals). Training consisting of tracing with a pencil inserted in grooves outlining each of the three shapes did not result in improved drawings. Thus, it is an adequate discrimination of parts of a figure which seems to be a prerequisite for the ability to draw it.

The present study was designed to investigate certain issues raised by Piagetian and related research concerning the development of children's notions of space. Specific hypotheses were:

Hypothesis 1: The order of difficulty of tasks will be haptic-visual discrimination, construction and then drawing.
Hypothesis 2: The smallness of the shape will facilitate its haptic identification, leading to considerable success by two and three year olds.
Hypothesis 3: The order of difficulty of all tasks according to shape will be circle, square, triangle, and diamond.
Hypothesis 4: Performance will be higher for children of enriched environments.

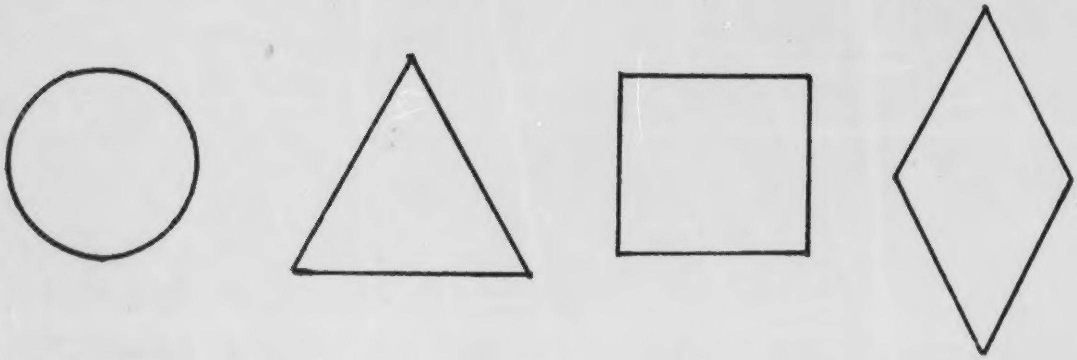
Method

Subjects. The sample population of 96 children was stratified by age and background. The four levels of age were two, three, four, and five years. The two levels of background were Background A: upper-middle class/half-day school/Montessori preschool and Background B: middle-class/day care/traditional preschool. The Montessori classroom environment is a structured one in which the students choose individual activities and exercise considerable autonomy. The traditional classroom environment has some individual activities, but includes many more activities which are organized by the teacher and presented to a group of children at the same time. The number of activities in each kind of environment related to the tasks in this study is about equal, though the organized Montessori drawing activities (insets, tracing letters and numerals, etc.) may facilitate drawing ability more than the freer drawing activities in the traditional environment. However, because the two and three year old and about one-third of the four and five year old Montessori children were new to the Montessori environment, this sample does not constitute a good Montessori/non-Montessori comparison. Another difference between the populations concerns the mothers' working habits. The mothers of the Background A sample generally did not work; those of the Background B sample did. The complicated nature of the background variable reflects reality rather than intentional choice -- traditional half day schools do not enroll two and five year olds and few upper-middle class mothers work. All of the children in the study were from a racially and economically heterogeneous community of 80,000 bordering on the north side of Chicago.

Design. Each S performed the haptic, construction, and drawing tasks and was re-interviewed six months after the first interview. Thus the data are cross-sectional and longitudinal. The independent variables in this study were Age (2, 3, 4, and 5 years), Background (A and B), and Time of Interview (Initial and Follow-up). The dependent variables were the scores for each shape on the haptic, construction, and drawing tasks.

Materials. The geometric shapes used in this study were constructed from wood 9mm. thick. Figure 1 shows the exact dimensions of each shape. The shapes fit into the palm of S's hand so that S could feel the entire shape simultaneously.

Figure 1



Four sets of shapes were used in this study, one by each experimenter. In each set, one shape was painted blue, one red, one green, and one yellow. The color of each shape was varied over sets.

The screen used in the haptic task was made of rigid cardboard 48 cm. by 48 cm. Holes were cut out of the screen and cloth sleeves were attached to the reverse side of the screen so that as S put his arms through the holes and sleeves, he could not see his hands. The screen was placed in front of S, who was at all times seated at a table. S could not see over the top or sides of the screen.

Two types of sticks were used in the construction task. The large sticks were wooden coffee stirrers 1 mm. thick, 6 mm. wide, and 9 cm. long. The small sticks were wooden matchsticks 5 cm. long from which the heads had been cut off.

The drawing was done with red pencils 10 cm. long on unlined white 8½" by 11" paper. Over the course of the experiment, it became clear that the lead in these pencils did not mark as well as it should. A #2 lead pencil would have given darker drawings.

Interviewers. The initial interviewing was done by a professor of early childhood, by a graduate student in educational psychology with some experience with children, and by an undergraduate with little experience with young children. The follow-up interviewing was done by the graduate student and by four undergraduates who had some experience with children.

The interviewers were trained in two sessions which included individual practice with children of the age sampled in the main study. Training was considered complete when both the trainer and an interviewer felt that the interviewer had moved through a complete interview easily and comfortably.

Interviewers were assigned to not more than one-half of each Age X Background cell so that in no case was an entire cell interviewed by one person.

Procedure. In each school, the testing was conducted in a quiet room or alcove familiar to the children. Ss were tested one at a time. For all children, the Haptic Task was given first, followed by the Construction Task and then the Drawing Task. Each interview lasted between ten and twenty minutes.

The Haptic Task: After seating herself and the child at the table, E pointed to the four geometric shapes laid out on the table in the order: circle, triangle, square, and diamond. S was asked to name as many of the shapes as he could. These names were recorded. The screen was then placed on the table, leaving the four sample shapes in the child's view, and S put his hands through the sleeves of the screen.

He was given his first shape in his right hand and asked which shape he held. If S had previously been unable to name the shape, he was asked to pull one hand out of the screen and point to the appropriate shape. Many younger Ss displayed a very powerful urge to pull the shape from behind the screen in order to look at it. E was sensitive to this and was at all times ready to hold on to the S's hand to prevent him from viewing the shape. The interview had been introduced to the children as a "secret"game, and most did not seem to mind this restraint. If S did not actively manipulate the shape in his hand and gave no response, he was encouraged by E to "touch it all over and feel it." S was also encouraged to make a choice even if he was unsure which shape was the correct one.

For each subject, the four shapes were presented twice in the same order. The order of presentation was randomized across subjects. The response to each shape and the method of response (naming, pointing) were recorded. Because the sample shapes were in view at all times and no time delay existed between shape presentation and identification, it is clear that the haptic task was in no way a memory task.

The Construction Task: Following completion of the haptic task, the screen was removed from S's view. Either a triangle or a square was placed on the table. S was given six large sticks and asked to build the given shape. Half of the subjects built the triangle first, and half the square first. The other of these two shapes was given second. The diamond was always presented last because a pilot study indicated that it was considerably more difficult to build than the other two, and it discouraged subsequent responses on the simpler shapes if presented first. The circle obviously could not be used in this construction task. If S could not construct the shape with the large sticks, the small sticks were given to him. E recorded S's responses by drawing on a record sheet all of S's attempts to construct each shape. Each side was labeled to indicate the order in which it was placed.

The Drawing Task: After completing the construction task, S was given a red pencil and a sheet of paper. S was presented each shape and asked to draw it on his paper. The circle was presented first, the diamond last, and triangle and square second or third, alternating between subjects. In the event that S's drawing of a particular shape was inaccurate, E presented that shape for S to redraw after S had attempted to draw each of the four shapes. S was also allowed to redraw a shape when he asked to do so. The name of each shape was recorded beside each drawing.

Scoring. The haptic task was scored for each shape by the total correct recognitions out of the two trials.

The recordings of the construction task for the square, triangle, and diamond were scored according to the following criteria:

0: not even one vertex was constructed correctly,
1: at least one vertex was constructed correctly but the entire shape was not constructed correctly,
2: at least one stick was initially misplaced but the shape was eventually correct (i.e., trial and error correct construction),
3: the shape was constructed correctly with no misplaced sticks during construction.

Each shape constructed by each S was given a score of 0, 1, 2, or 3. All constructions were scored by two scorers. Disagreements were discussed and mutually resolved. The scorers initially agreed on 95% of the constructions.

The drawing of each shape was scored by six judges on a similar 0-3 point scale. The scale was modified somewhat for each shape as follows:

0: C: random scribble,
S, T, D: random scribble or a closed curve,
1: C: a score of 1 was not given for a circle,
S, T, D: distinct sides present, but some of the sides were curvilinear instead of rectilinear,
2: C: closed but not round; must be a distinct closed curve and not a circular scribble,
S, T: all sides approximately straight; all vertices clearly present; angles or sides may be of unequal size,
D: two halves of the diamond are present but are not coordinated correctly (lines joining opposite vertices are not perpendicular), or the sides are not equal,
3: C: fairly round,
S, T: all sides quite straight; no extensions at the vertices; angles and sides all equal (one side $\leq$ 3/2 of another)
D: lines joining opposite vertices are perpendicular, all sides are equal, and angles are the correct size.

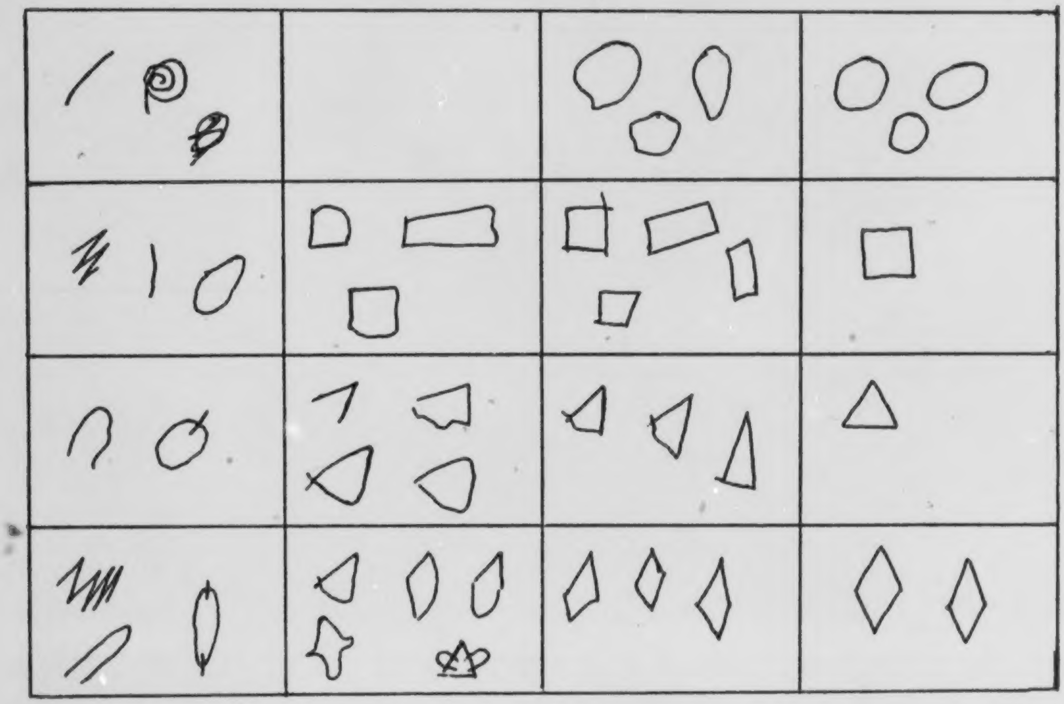
Classifying drawings is a difficult task. There are some fairly clear stages present in the drawing of these shapes, and the rating scale was constructed to describe these stages. Figure 2 illustrates some of the typical drawings in these stages. However, many drawings fall in between these stages; that is, the drawings show continuous rather than discrete development. This continuity is reflected in the ratings of the drawings. Table 1 displays the percent agreement among the six raters for each shape in each cell. The agreement fell as children moved from one stage to another. Many of the Background A three year olds were between stage 0 and 1 for the last three shapes, while most Background B three year olds were more clearly

Table 1

Percent Agreement of Drawing Ratings by Six Judges for Each Shape by Age/School Sample

Age	Back-ground	C	S	T	D	Mean
3	B	81.9	75.0	80.6	87.5	82.7
3	A	81.9	70.1	77.8	77.8	76.9
4	B	87.5	77.8	83.3	79.2	82.0
4	A	86.1	88.9	90.3	90.3	88.9
5	B	72.2	88.9	83.3	84.7	82.3
5	A	80.6	76.4	83.3	86.1	86.1
Mean		81.7	79.5	83.3	84.3	82.4

Figure 2



in stage 0. The agreement was higher for the latter group. The "A" four year olds were quite solidly in stage 2 for the square, triangle and diamond, and agreement was fairly high (88.9%). The agreement fell for the "A" five year olds because many of these Ss gave responses that fell between scores 2 and 3, some raters giving them a 2 and some a 3. Lower agreement for the five year olds was also due to what we term "the sloppy effect". Some children become quite casual about shapes they think are easy, and they produce drawings with characteristics of lower scores. While it was fairly clear that some of the drawings could have been done at a higher level, they had to be scored as they were - somewhere between stages 2 and 3. This depressed the agreement. In future studies, researchers should be urged to ask Ss to re- and re-draw a shape if it was drawn quite rapidly or sloppily.

In order to use this continuous rather than discrete quality of the scoring for the drawings, the scores of all raters for a particular shape were totalled. These totals were used in all of the analyses of the drawings.

Analysis

A 2 X 3 multivariate analysis of variance was performed on the haptic, construction, and drawing scores for the initial and follow-up interviews. The independent variables were Background (A and B) and Age (3, 4, and 5 years old). The variates were the scores awarded for each shape (circle, square, triangle and diamond). A 2 X 2 X 3 MANOVA was also performed on the scores comparing performance on each interview. Here the independent variables were Time (first and second interviews), Background, and Age. Two Ss were lost in the second interview because they had moved from the area. The MANOVA used was originally written at the Biometric Laboratory, University of Miami, Coral Gables, Florida.

Extreme difficulties were encountered in our interviews with two-year-old Background B children. Communication was difficult, and Ss' attention spans were too short to reach task completion. Because it is misleading to compare incomplete task performance, the results for each two-year-old Background cell were analyzed apart from the remaining sample data. Descriptive data for the two year olds is given in a later section.

In order to determine whether the ability to perceive haptically, to construct, and to draw shapes occurred in individual Ss in the predicted order (circle first, then square, then triangle, then diamond), the number of observed reversals of this order within each S was counted for each task. A reversal occurred whenever the score for a given shape was less than that for a shape hypothesized to be more difficult. Thus, a sequence of scores such as 3, 1, 1, 2 for the circle, square, triangle, and diamond, respectively, was counted as two reversals. In this example, both the square and triangle scores were less than the diamond score. A chi square was calculated for each Background X Age cell comparing observed reversals to those which should have occurred by chance alone.

Results

Haptic Perception Task

Initial Interview: The means and standard deviations for the initial interview haptic task are given in table 2. The percent of correct responses are given by each half year in table 3. Virtually all of the children correctly identified the circle. Except for the diamond for the four year old B children, the four and five year olds were almost entirely correct in their identifications. Most of the errors in the three year olds came from children less than three years, six months old. So, in this sample, most of the children were able to complete the haptic task successfully after three and a half years of age.

Table 2

Means and Standard Deviations of Correct
Scores for the Haptic Task (Initial Interview)

			Shapes			
Age	Background		Circle	Square	Triangle	Diamond
3	B	$\bar{X}$	2.000	1.750	1.667	1.667
		SD	0.000	.622	.651	.778
3	A	$\bar{X}$	1.917	1.917	1.750	1.583
		SD	.289	.289	.622	.669
4	B	$\bar{X}$	2.000	1.833	2.000	1.750
		SD	0.000	.577	0.000	.622
4	A	$\bar{X}$	2.000	1.917	2.000	1.917
		SD	0.000	.289	0.000	.289
5	B	$\bar{X}$	2.000	2.000	2.000	2.000
		SD	0.000	0.000	0.000	0.000
5	A	$\bar{X}$	2.000	2.000	2.000	2.000
		SD	0.000	0.000	0.000	0.000

scores: 2: 2 out of 2 trials correctly identified
1: 1 out of 2 trials correctly identified
0: 0 out of 2 trials correctly identified

The multivariate analysis of variance revealed no significant interactions, no effect of Background, and no effect of Age (though the latter approached significance with $p < .06$). The univariate F tests for Age were significant for the triangle ($F = 5.04$, $p < .009$) and for the diamond ($F = 3.34$, $p < .042$). Younger children made more incorrect identifications of these shapes.

Table 3

Percent Correct Haptic Perceptions By Full and Half-Year Cells

First Interview

		Shape						Shape			
Age	(n)	C	S	T	D	Age	(n)	C	S	T	D
3.0-3.6	(13)	96	92	77	69						
3.7-3.11	(11)	100	91	95	95	3.0-3.11	(24)	98	92	85	81
4.0-4.6	(12)	100	96	96	96						
4.7-4.11	(12)	100	92	100	88	4.0-4.11	(24)	100	94	98	92
5.0-5.6	(15)	100	100	100	100						
5.7-5.11	(9)	100	100	100	100	5.0-5.11	(24)	100	100	100	100

Second Interview

		Shape						Shape			
Age*	(n)	C	S	T	D	Age*	(n)	C	S	T	D
3.0-3.6	(11)	100	100	100	100						
3.7-3.11	(11)	100	91	100	91	3.0-3.11	(22)	100	96	100	96
4.0-4.6	(12)	100	100	96	100						
4.7-4.11	(12)	100	100	100	100	4.0-4.11	(24)	100	100	98	100
5.0-5.6	(15)	100	100	100	100						
5.7-4.11	(9)	100	100	100	100	5.0-5.11	(24)	100	100	100	100

scores: 3: immediately correct construction
2+3: trial and error or immediately correct construction

* The children were now six months older than the ages listed.

Follow-Up Interview: In the first interview most of the haptic errors had been made by children younger than three and a half. This result was confirmed in the second interview with the same children six months later -- only six errors were made in the 560 haptic identifications. Two square and two diamond errors were made by three year olds, one diamond error by a four year old, and one triangle error by a five year old. The haptic identification of small shapes reaches a quite dramatic ceiling at the age of three and one-half.

Construction Task

Initial Interview: In table 4, the group means for the construction task reveal that for three and four year olds, the square was considerably easier to construct than the triangle, and the triangle was considerably easier than the diamond. For the five year olds, the square and triangle reached ceiling performance while the diamond still presented some difficulties.

In table 5 the percent correct responses indicate that 82% of the three and one-half year old Ss correctly constructed the square either by trial and error or without hesitation. Only 75% of the four and one-half year old Ss were able to construct the triangle, and only 83% of the five and one-half year olds were able to construct the diamond.

The multivariate analysis of variance revealed no significant interaction and no significant effect of Background overall, though for the diamond, Background A children made significantly higher scores ($F = 4.79$, $p \angle .032$). The overall effect of Age was significant ($F = 8.151$, $p < .0001$), as were the univariante F tests for all three shapes ($p < .001$): Square, $F = 9.45$, Triangle, $F = 25.03$; Diamond, $F = 17.42$. Thus construction ability for each shape increased with age.

Follow-Up Interview: The percent correct performance on the second interview construction task is reported in table 5. These data reveal that almost the entire sample was able to construct the square and the triangle correctly either immediately or by trial and error (above 90% correct for all age levels). This is considerably above the performance on the first interview, for at that time even the four and one-half year olds were only 75% correct on these shapes. The teachers of the children reported that they had not tried to teach these constructions to the children. It may be that the first interview focused the attention of the children on the construction of shapes, and they subsequently set about to learn this task for themselves. If this is so, it would seem that a task interview is an extraordinarily powerful means of motivating children to learn this task.

Table 4

Means and Standard Deviations of Scores For the Construction Task (Initial Interview)

Age	School		Square	Shapes Triangle	Diamond
3	B	$\overline{X}$	1.667	1.083	.583
		SD	1.303	.900	.515
3	A	$\overline{X}$	2.167	1.417	.750
		SD	1.030	.900	1.055
4	B	$\overline{X}$	2.417	2.000	1.250
		SD	.996	1.128	.965
4	A	$\overline{X}$	2.667	2.167	1.750
		SD	.778	1.030	.965
5	B	$\overline{X}$	3.000	2.917	1.833
		SD	0.000	.289	.937
5	A	$\overline{X}$	3.000	3.000	2.545
		SD	0.000	0.000	.688

scores: 0: not even one vertex
1: one vertex correct
2: trial and error correct
3: immediately correct

Table 5

Percent Correct Constructions
By Full and Half-Year Cells

First Interview

			Shape					Shape	
Age	Score	S	T	D	Age	Score	S	T	D
3.0-3.6	3	15	08	0					
n=13	2+3	54	15	08					
3.7-3.11	3	73	18	09	3.0-3.11	3	46	13	04
n=11	2+3	82	45	18	n=24				
4.0-4.6	3	67	42	17					
n=12	2+3	83	58	33					
4.7-4.11	3	75	58	17	4.0-4.11	3	71	50	17
n=12	2+3	83	75	67	n=24	2+3	83	67	50
5.0-5.6	3	100	93	40					
n=15	2+3	100	100	73					
5.7-5.11	3	100	100	50	5.0-5.11	3	100	100	43
n=9	2+3	100	100	83	n=24	2+3	100	100	78

Second Interview

			Shape					Shape	
Age*	Score	S	T	D	Age*	Score	S	T	D
3.0-3.6	3	73	54	09					
n=11	2+3	91	91	27					
3.7-3.11	3	64	45	0	3.0-3.11	3	68	50	04
n=11	2+3	100	91	36	n=22	2+3	95	91	27
4.0-4.6	3	92	50	17					
n=12	2+3	100	92	33					
4.7-4.11	3	92	83	50	4.0-4.11	3	83	42	33
n=12	2+3	100	92	92	n=24	2+3	100	92	63
5.0-5.6	3	100	100	73					
n=15	2+3	100	100	100					
5.7-5.11	3	100	89	67	5.0-5.11	3	100	96	71
n=9	2+3	100	100	100	n=24	2+3	100	100	100

Scores: 3: immediately correct construction
2+3: trial and error or immediately correct construction

*The children were now six months older than the ages indicated.

The number of immediately correct responses varied by shape and by age. At ages three and four, 20% more constructions were immediately correct for the square than for the triangle. Both shapes were constructed immedately by five year olds. The diamond was eventually correctly constructed by almost all of the children who were now five and older. However, even one-third of the oldest children (now six to six and one half) still required trial and error to construct the diamond correctly.

The multivariate analysis of variance results were the same as for the first interview.

Comparison of the Two Interviews: A 2 X 2 X 3 (Time X Background X Age) multivariate analysis of variance for construction revealed significant effects ($p < .001$) for Time and Age overall and for each shape. Older Ss and Ss in the second interview made higher scores. The Background effect for the diamond almost reached significance ($F = 3.49$, $p < .064$), with Background A children scoring consistently higher on the diamond construction. There was a significant Time X Age interaction overall ($F = 2.25$, $p < .04$) and for the triangle ($F = 5.07$, $p < .008$). This interaction just failed to reach significance for the square ($F = 2.62$, $p < .07$). For both the square and triangle, the three year olds increased their scores in the follow-up interview about twice as much as did the four year olds, and the fives did not increase. In contrast, the increase for the diamond showed a regular pattern across ages, with Background A children increasing about .6. No other effects reached significance.

Drawing Task

First Interview: In table 6, the group means for the drawing task reveal that children scored the highest on the circle, followed by the square, triangle, and diamond, in that order. The percent correct scores given in table 7 show the irregular growth of drawing abilities specific to the individual shapes. Even at the age of five, most Ss did not make drawings that were correct by the rather rigorous standards of the drawing rating scale.

The multivariate analysis of variance showed significant effects for Age overall and for each shape on the univariate tests ($p < .001$). Drawing performance improved with age for each shape. The overall effect of Background was not significant, but the univariate tests showed a significant Background effect for the circle ($F = 4.49$, $p < .038$) and the diamond ($F = 5.05$, $p < .028$), while the square ($F = 2.91$, $p < .093$) and the triangle ($F = 2.29$, $p < .135$) approached significance. Background A subjects scored higher than B subjects. There was a significant overall Interaction of Background and Age, though the circle was the only shape to show significance on the univariate test. This interaction was a result of A children scoring higher on the drawing of the circle and the square at younger ages and higher on the triangle and diamond at the older ages.

Table 6

Means and Standard Deviations of Scores for the Drawing Task

First Interview

			Shapes			
Age	Back-ground		Circle	Square	Triangle	Diamond
3	B	$\bar{X}$	1.864	.667	.576	.409
		SD	.446	.500	.668	.513
3	A	$\bar{X}$	2.394	1.182	.758	.455
		SD	.382	.497	.375	.342
4	B	$\bar{X}$	2.431	1.528	1.111	.639
		SD	.411	.497	.609	.703
4	A	$\bar{X}$	2.444	1.556	1.375	1.236
		SD	.385	.679	.782	.851
5	B	$\bar{X}$	2.514	1.778	1.750	1.472
		SD	.305	.583	.680	.688
5	A	$\bar{X}$	2.542	1.958	1.889	1.833
		SD	.421	.370	.239	.555

Second Interview

			Shapes			
Age	Back-ground		Circle	Square	Triangle	Diamond
3	B	$\bar{X}$	2.439	1.182	.995	.758
		SD	.319	.450	.587	.664
3	A	$\bar{X}$	2.394	1.697	1.303	.697
		SD	.436	.623	.714	.862
4	B	$\bar{X}$	2.306	1.597	1.444	1.181
		SD	.388	.575	.574	.712
4	A	$\bar{X}$	2.611	1.958	1.917	1.694
		SD	.372	.276	.584	.413
5	B	$\bar{X}$	2.667	1.958	2.250	1.889
		SD	.302	.578	.447	.259
5	A	$\bar{X}$	2.819	2.056	2.125	2.000
		SD	.181	.287	.285	.123

Scores: 0: meaningless scribble
1: one vertex correct
2: whole shape present but still some errors
3: shape correct in all details

Table 7

Percent Correct Drawings
By Full and Half-Year Cells

First Interview

		Shape						Shape			
Age	Score	C	S	T	D	Age	Score	C	S	T	D
3.0-3.6	3	31	0	0	0						
n=13	2+3	92	08	0	0						
3.7-3.11	3	27	0	0	0	3.0-3.11	3	29	0	0	0
n=11	2+3	83	09	18	0	n=24	2+3	87.5	08	08	0
4.0-4.6	3	25	0	0	0						
n=12	2+3	100	42	25	08						
4.7-4.11	3	50	0	0	0	4.0-4.11	3	42	0	0	0
n=12	2+3	100	75	67	58	n=24	2+3	100	58	46	33
5.0-5.6	3	60	0	07	0						
n=15	2+3	100	87	73	60						
5.7-5.11	3	67	33	11	22	5.0-5.11	3	62.5	12.5	08	08
n=9	2+3	100	89	100	67	n=24	2+3	100	87.5	83	62.5

Second Interview

		Shape						Shape			
Age*	Score	C	S	T	D	Age*	Score	C	S	T	D
3.0-3.6	3	54	0	0	0						
n=11	2+3	100	64	27	18						
3.7-3.11	3	45	0	0	0	3.0-3.11	3	50	0	0	0
n=11	2+3	100	45	27	27	n=22	2+3	100	54	27	23
4.0-4.6	3	33	0	08	0						
n=12	2+3	100	67	58	25						
4.7-4.11	3	75	0	08	0	4.0-4.11	3	54	0	08	0
n=12	2+3	100	92	83	75	n=24	2+3	100	79	71	50
5.0-5.6	3	80	13	27	0						
n=15	2+3	100	93	100	93						
5.7-5.11	3	100	11	22	0	5.0-5.11	3	87.5	12.5	25	0
n=9	2+3	100	78	89	100	n=24	2+3	100	87.5	96	96

Scores 3: immediately correct drawing
2+3: trial and error or immediately correct drawing

*The children were now six months older than the ages indicated.

Table 8

Multivariate Analysis of Variance for the Drawing Task

	Multivariate Analysis		Univariate Analysis				
Main Effects	df	f	df	Circle F	Square F	Triangle F	Diamond F
Time	4,125	5.29^{**}	1,128	7.58^{**}	11.47^{**}	19.21^{**}	12.92^{**}
Background	4,125	3.74^{*}	1,128	6.76^{*}	10.37^{*}	$4.86^{\#}$	7.05^{*}
Age	8,250	11.66^{**}	2,128	11.15^{**}	25.88^{**}	43.39^{**}	47.48^{**}
Interactions							
T X B	4,125	<1	1,128	<1	<1	<1	<1
T X A	8,250	1.37	2,128	1.64	1.68	<1	<1
B X A	8,250	$2.29^{\#}$	2,128	<1	1.81	1.28	2.56
T X B X A	8,250	1.59	2,128	4.43^{*}	<1	<1	<1

#: $P<.05$, *: $P<.01$, **: $P<.001$

Table 9

Number of Reversals by Individual Children of the Order Square, Triangle, and Diamond

	Haptic Task				Construction Task				Drawing Task			
	Background B		Background A		Background B		Background A		Background B		Background A	
Age	No. of reversals	X^2	No. of reversals	X^2	No. of reversals	X^2	No. of reversals	X^2	No. of reversals	X^2	No. of reversals	X^2
Interview 1												
3	3	7.41^{**}	4	6.31^{*}	1	10.67^{**}	3	7.41^{**}	2	9.30^{**}	2	9.30^{**}
4	4	6.31^{*}	2	9.30^{**}	1	10.67^{**}	0	12.52^{**}	2	9.30^{**}	1	10.67^{**}
5	0	12.52^{**}	0	12.52^{**}	0	12.52^{**}	0	12.52^{**}	4	6.31^{*}	3	7.41^{**}
Interview 2												
3	0	12.52^{**}	4	6.31^{*}	0	12.52^{**}	0	12.52^{**}	1	10.67^{**}	0	12.52^{**}
4	0	12.52^{**}	0	12.52^{**}	3	7.41^{**}	0	12.52^{**}	1	10.67^{**}	2	9.30^{**}
5	0	12.52^{**}	1	10.67^{**}	0	12.52^{**}	0	12.52^{**}	4	6.31^{*}	0	12.52^{**}

There were 36 possible reversals in each cell. 13.84 would occur by chance.

* $P<.05$ ** $P<.01$

Second Interview: All children received scores of 2 or 3 on the circle. At age three, the percent of children receiving scores of 2 or 3 was 50% for the square and about 25% for the triangle and diamond (see table 7). At age four, this percent was about 75% for the square and triangle, and 50% for the diamond. By age five, almost all children received scores of 2 or 3 on all drawings. However, only half of the three and four year olds received a 3 for a circle, while most of the five year olds did. Very few of the children of any age received a 3 on drawings of the square, triangle, or diamond.

As in the first interview, Age was a significant effect overall and for each shape ($p < .001$); older children had higher drawing scores. Background was not a significant effect overall, but it was for certain shapes. The A children were significantly better in drawing the square ($F = 3.08$, $p < .008$), with trends in superiority for the other shapes (triangle: $F = 3.08$, $p < .084$; diamond: $F = 2.11$, $p < .151$). The interaction between Age and Background was not significant in the second interview.

Comparison of the Interviews: The 2 X 2 X 3 multivariate analysis of variance of scores on the drawing task indicated that each of the main effects - Age, Background, and Time of interview - was significant overall and for each shape (see table 8). Performance was higher for older Ss, for the Background A children, and for the follow-up interview.

The significant Background effect indicates the complexity of the developmental growth for drawing different shapes. This development seems to proceed in stages, with fairly marked improvements of particular types followed by long plateaus of little or no improvement. These periods of marked improvement differ for different shapes. For all shapes, A children preceded the B children in their movement to a new stage, but some of this movement took place at the time of the first interview and some, only by the time of the second. Thus, Background A children scored significantly better on the circle and the diamond at the first interview and on the square at the follow-up interview.

The Age X Background interaction was significant overall in the comparison MANOVA. This interaction indicated that the difference in scores between the A and B children varied at different ages and that this difference also varied with each shape. For the square, the A children were considerably superior at age three and only moderately so at age four and five. For the triangle, the A children were moderately superior at age three and four, and not superior by age five. For the diamond, there was no difference at age three, but the A children scored considerably higher at age four and five. Background A children seemed to score higher when the ability to draw a shape began to develop.

The three-way interaction (Age X School X Time) was significant for the circle ($p < .01$). The three year old A children were superior on the circle drawing for the first interview but not for the second; the four and five year old A children were superior on the circle in the second interview and not in the first. We believe the A children were probably superior at the older ages in the first interview, too, but the "sloppy effect" (older children not drawing the simple shapes as carefully as they could) pulled down scores. It is probable that the scores were affected more on the first than the second interview because experimenters were less sensitive to this sloppiness problem at that time and asked for fewer re-drawings than in the second interview.

Comparison of the Three Tasks

A comparison of the percent correct haptic-visual identifications, constructions, and drawings in tables 3, 5, and 7 reveal that while haptic identifications for all shapes reached ceiling performance by 3.6 years, construction and drawing for certain shapes lagged considerably behind. Using a criterion of 66% correct, we find that, for the first interview, trial and error construction (scores 2 and 3) of the square lagged six months behind haptic identification, and trial and error construction of the triangle and diamond lagged one and a half years behind haptic success. Approximate drawing success (scores 2 and 3) was one year behind trial and error construction for the square and diamond, but occurred at the same time as construction for the triangle.

A comparison of scores of 3 (immediately correct construction and entirely correct drawing) shows that construction success lagged behind haptic success from six months to over three years, varying from the square to the diamond. A very considerable construction-drawing lag exists using scores of 3. An immediately correct construction was made by 66% of the children by age 3½ for the square, by age 5 for the triangle, and by age 6½ for the diamond; but even by age 6½, the percentages of children who had made entirely correct drawings was only 11% for the square, 22% for the triangle, and 0% for the diamond.

Comparison of the Shapes

The number of reversals of the order square, triangle, diamond observed in the protocols of individual Ss and chi-square values for these reversals appear in table 9. For all tasks in both interviews the number of reversals is low and significantly below chance. Thus, for individual children as well as for groups, the circle was the easiest to identify haptically, to construct and to draw, followed by the square, then the triangle, and then the diamond.

Two-Year Performance

It was necessary to delete the two year old data from our main analyses because so few children in the B sample were able to complete all of the tasks. No children of this age were able to construct or draw the geometric shapes, with the exception of one female Background A child, age 2:07, who was able to construct the square without hesitation.

Striking differences were observed between the A and B subjects. The B Ss were from poverty homes and attended a day-care center which provided for some physical, emotional, and motor skill development but gave little contingent verbal interaction. None of the children gave names for the geometric shapes, and only four of the twelve sampled could or would complete the haptic task. In contrast, all of the A children completed our tasks, and five out of the twelve could name the circle, square and triangle. None of these children could name the diamond.

In the initial interview it was unclear whether the B children could not perform our tasks or whether the interviewers were unsuccessful in communicating the tasks to them. Fear and shyness did seem to be a factor in their low performance. Consequently, after the follow-up interview, some children were re-interviewed in their classroom using a towel-sleeve for hiding the shapes, rather than the experimental screen. Half of the children improved their performance under these conditions. It is not clear which of the following variables produced this improvement: familiar classroom, seeing other children do the task, the towel-sleeve instead of the screen, or a second trial. Perhaps the combination was important. What does seem clear is that testing conditions are especially important for young low SES children. Other investigators are encouraged to take steps to reduce Ss' apprehension concerning the experimental environment and to increase non-verbal communication of the required tasks.

In the six month follow-up interview the A Ss completed almost all of the haptic tasks correctly (there were two square and two diamond errors). Thus by two and one-half years of age, these A children performed as well as did the three and one half year olds in the total sample. Whether this is due to some aspect of their Montessori training is unclear (they attended for one and a half hours twice a week); their curriculum did not involve tasks similar to the haptic task. Perhaps the fact that these very young children were enrolled in this preschool reflected a special interest or knowledge on the part of their parents which contributes in other ways to the advancement of their child's cognitive abilities. Alternatively, perhaps what these two year olds brought from their Montessori activities was an increased ability to focus on a task.

The two year old data does support the hypothesis concerning the order of difficulty of particular shapes in the haptic task. There were extremely few (three out of a possible 144) reversals of the order circle > square > triangle > diamond. The percent of correct haptic recognitions for all two year olds is reported in table 10.

Table 10

Percent Correct Recognition by Two Year Olds
Of Shapes in the Haptic Task

	First Interview					Second Interview				
	n	C	S	T	D	n	C	S	T	D
Background B	12	21	4	8	0	8	81	69	44	0
Background A	12	88	54	90	25	9	100	89	100	89

Secondary Results

Names: An analysis of Ss' knowledge of the names of each shape showed that by three years of age over 60% of all Ss could name the circle, square and triangle. By five years of age virtually all Ss could name these shapes. However less than 5% of the three year olds could name the diamond. Only 17% of the four year olds and 71% of the five year olds knew the name of the diamond. Knowledge of the shape names was substantially higher in the second interview. Ninety one percent of the three year olds knew the names circle and square: 86% knew the name triangle. The diamond was named by only 32% of the three year olds; however 54% of the four year olds and 96% of the five year olds could name the diamond. This relatively late acquisition of the names of the shapes demonstrates that the ability to name a shape was not a prerequisite for the haptic identification of a shape, as haptic identifications of all shapes were performed by age three and one-half.

Matchsticks and Coffee-stirrers: The percentage of Ss at each age who were not able to construct each shape with coffee-stirrers but could do so with the smaller matchsticks was calculated. In the first interview, the matchsticks facilitated construction of the square for 18% of the three and 33% of the four year olds; for the triangle 42% of the three and 10% of the four year olds were aided; and 9% of the three year olds, 17% of the four year olds, and 14% of the five year olds were able to construct the diamond only when given the matchsticks.

There were fewer Ss in the follow-up interview than in the first who were unable to construct the shapes. However, of those Ss unable to construct the shapes, a larger percentage were aided by the matchsticks; 50% of the three year olds and the one four year old unable to construct the square with the large sticks were able to construct it with the matchsticks. For the triangle, 50% of the three year olds and 33% of the fours were aided by the matchsticks; and 16% of the three and 10% of the four year olds needed the matchsticks in order to construct the diamond.

Patterns in Construction: We examined the order in which each stick-side of each shape was placed in an effort to find any consistent patterns for form construction. All figures given are percents of the successful constructions (score 2 or 3) at a given age level. The most consistent pattern observed for the square was a circular placement in which sticks were placed in a clockwise or counter-clockwise order by over 60% of the Ss at each age. The most consistent pattern observed in the triangle constructions was one in which the two diagonal sides were placed first, followed by the bottom, horizontal side. This pattern was observed in over 70% across all age levels and reached 83% for the five year olds. The diamond constructions were more varied than the other two. The diamond was constructed in a contiguous circular (clockwise or counter-clockwise) fashion by 25% of the three and four year olds and by 61% of the five year olds. However 50% of the three, 67% of the four, and 33% of the five year olds constructed the diamond in top/bottom halves, placing the top sticks left to right (or right to left) and then the bottom sticks left to right (or right to left, accordingly). This indicates a considerable number of Ss who produced (that is, displayed a knowledge of) a symmetrically balanced diamond.

The follow-up construction data revealed somewhat different patterns. The square was constructed in a contiguous circular (clockwise or counter-clockwise) fashion by 30% of the three, 67% of the four, and 29% of the five year olds--a striking decrease from the over 60% observed in the initial interview for all ages. However, 32% of the fours and 54% of the fives constructed the square by placing the first two sides simultaneously. One of the fours and six five year olds who displayed this tendency intially placed sticks which formed the parallel sides of the square. However, five of the six fours and seven of the thirteen five year olds immediately placed two sticks to form a vertex of the square. This finding combined with those Ss who constructed the square in a circular fashion (consistently placing sticks which formed successive vertices) implies that it was the vertex which was the most salient feature of the square for those Ss who were able to construct the square (rather than the parallel sides, for example). In the construction of the triangle, there was an increase from the first interview in the tendency to place the bottom horizontal stick last. Over 85% (95% for the fives) displayed this pattern. Also, over 32% of the three and four and 71% of the five year olds simultaneously placed the diagonal sides of the triangle in their constructions. Certain tendencies found in the diamond construction in the initial interview were again found in the follow-up interview. Forty-three percent of the threes constructed the diamond in a circular order; that number dropping to 13% for the fours and only 4% for the five year olds. The tendency to construct the diamond in top and then bottom halves was observed for 43% of the threes, 67% of the fours and 83% of the five year olds. A lesser number (fours: 20%, fives: 8%) constructed the diamond in a left/right (or right/left) fashion. It is interesting to note that 47% of the fours and 67% of the fives simultaneously placed at least two sides of the diamond before constructing the rest of the shape. Thus, over 50% of our sample at each age who were able to construct the diamond displayed an understanding of the symmetrical nature of the diamond through their method of construction.

Discussion

Our hypothesis that children would be able to perceive geometric shapes haptically prior to constructing them, and to construct them prior to drawing them, was strongly supported. Virtually all of the children could haptically identify the circle, square, triangle, and diamond by the age of 3.6 years. Over two-thirds of the children could construct the square by age 4.0, the triangle by age 5.0 and the diamond by age 6.0. However, even at 6.0 years, less than one-third of our Ss were able to draw any shape other than the circle quite accurately.

Our results clearly support the hypothesis that the tasks involving the circle are solved first, followed by those involving the square, then the triangle, and finally the diamond. This was true for haptic, construction, and drawing tasks. The oblique nature of the sides of the triangle make it more difficult than the square, and the simultaneous coordination of the right-left and the top-bottom halves of the diamond makes this shape the most difficult.

It should be noted that progress in an individual child is not the smooth increase indicated by our group data. Individual progress consists of slow development of skills, sudden insight, and temporary regression. Regressions may occur as a child works on some new problem and temporarily ignores a previously solved one, as indicated by our four year olds when they abandoned equilateral squares with rounded corners to draw rectangles with 90^{o} vertices. Regressions also may be due to the state of the child at a given moment, involving uncontrollable internal variables such as hunger, illness or boredom.

Our results agree with some of Piaget and Inhelder's results and disagree with others. Piaget and Inhelder deliberately chose large figures for their haptic task because they were interested in the ability of children to coordinate perceptions of parts of figures. They found that young children (three years) did not seek out parts of figures and that older ones (four years) either did not seek out all of the parts or were unable to coordinate them. Their haptic task also involved a large number of shapes, many of which were quite similar to the square, triangle, and diamond. Our study indicated that when children did not have to search for parts of the figure, they could make haptic identifications by age three and one half. This result is in striking contrast to results with larger shapes, as for example Lovell who reported that only three out of 43 children between three and one half and four could identify the circle, square and triangle from a haptic presentation (Lovell, 1959, p. 107).

The chief difference between our results and those of Piaget and Inhelder is that construction clearly precedes drawing. It is interesting that this result is more in keeping with Piagetian theory in general than is Piaget's own result, for one would expect children to be able to operate on concrete objects before they could operate on abstract images. The drawing of shapes occurs later than constructions for several reasons.

Drawing does not allow for trial and error placements of a part; each mistake requires a whole new drawing. Drawing also requires the ability to draw a straight line. Most importantly, the drawing task involves the coordination of lines one has already drawn on paper with mental images of lines still undrawn which one projects onto the paper. As one five year old subject described it, "I see an invisible triangle on the paper, and I draw it." Drawing shapes is a difficult task, requiring a detailed and well-formed mental image as well as the ability to coordinate such images with partially completed drawings. Children did seem to use mental guides almost exclusively for drawing, for very few Ss were observed to look back and forth from the presented shape to their drawing. Even when interviewers called a child's attention to the shape while he was drawing, the child ignored it.

The construction task used in this study utilized sticks of equal length. We gave Ss six sticks, from which they chose the appropriate number to build each shape. It should be noted that it was only when Ss were unsure or unable to build the shapes that they tried to use all six sticks. That is, our Ss did display an understanding that each shape had an exact number of sides. However, we cannot be sure that they fully understood that each shape had sides of equal length. For example, there was no question that if our Ss placed three sticks together touching ends, they had constructed an equilateral triangle. Perhaps a better test of a child's understanding that an equilateral triangle, a square, and a diamond have equal length sides would be made if we had given Ss sticks of many lengths. When given sticks of many lengths, Ss would have to choose those sticks of equal length and then construct the shape. The difficulty in performing this new task should fall between that of the present study's construction task and drawing task. This prediction stems from the assumption that the difficulty involved in making equilateral sides with sticks should be the same for the suggested construction task and the present drawing task, but the former will involve the coordination of concrete materials rather than a mental image.

The ceiling performance reached on both the haptic and the construction tasks undoubtedly contributed to the lack of significance of the Background effect. Where there were differences, Background A scores were consistently higher. The one significant construction difference found was for the diamond - the most difficult task. That Background differences might have been found in the haptic task with younger children is supported by a study by Fuson and Love-Kunesh (1977). Using shapes identical to those used in this study, they found significantly better performance in the visual-visual and haptic-visual shape identification by middle-class children aged 1½ to 3 years whose mothers had fourteen years or more of education than by children whose mothers had between eleven and thirteen years of education.

In both the construction and drawing tasks, Background A children were the first to demonstrate an advance in skill for a particular shape, though the considerable difference which resulted occurred sometimes in the first interview and sometimes in the follow-up interview. The scale used to rate the drawings helped to minimize Background differences, for it combined a wide range of drawings in category 2. Background A children tended to make drawings that were not quite "3", while Background B children often made level "2" drawings that were barely past level "1". A more discriminating scale for the better drawings would have resulted in even larger differences between the two background groups.

Our interviewers did report additional differences between the Background A and B subjects, especially in the second interview. These impressions are offered here to aid the generation of hypotheses in future studies. The Background A children were generally more confident and self-possessed, attended more closely to the tasks in the interview, and made more verbalizations indicating they understood the overall structure of the interview ("We can't construct the circle with these sticks." and "And now you want me to draw the diamond.") These kinds of differences might be examined in future studies.

Drawings necessarily reflect both motoric and cognitive abilities. If one side of a square is drawn in a convex rather than a straight line, for example, it is difficult to decide whether the drawing is due to an inaccurate production scheme, to a lack of motoric control, or to an inaccurate mental image of the shape. Our scoring plan did not, and we believe could not, make a clear separation here. Freehand drawings will always confound these sources of error. A possible alternative is to offer drawing aids, such as a straight-edge and measuring instrument, to S while he is drawing. In this way one could observe how a child went about making his drawings. With young children, however, these unfamiliar aids may be of little use.

In order to assess the level of motoric skill tapped by our scoring scale, a mathematics class for preservice elementary teachers was asked to draw the shapes used in this study. They were allowed to draw and redraw each shape until they were satisfied with it (many of the older children in the main study had voluntarily done this as well). The number of three's (the highest score) for the seventeen students was circle: 17, square: 14, triangle: 11, and diamond: 6. The triangle scores of "2" resulted chiefly from inability to draw straight oblique lines; this resulted in the sides of the triangle being curved in various ways. The diamond scores of "2" were similar to the better two drawings of our five year old sample, displaying an inability to make the two halves of the diamond symmetric. Either the top and bottom, or the left and right sides did not match. Thus in this adult sample, as in the preschool sample, the lack of ability to coordinate two dimensions simultaneously precluded accurate drawing of the diamond. Again, the need for another level in our drawing scale to separate the wide range of drawings at level "2" was evident.

Several methodological problems arose in our use of an interview situation. Interviewing young children is inherently unreliable. Many internal (hunger, fatigue, illness) and external (the amount of threat in the situation, the activity the child just left, the presence of an unfamiliar adult) factors can affect a child's performance. We attempted to control for as many of these as possible: interviewing was always done at the beginning of school; we tried not to interrupt a child at work; a friend or teacher could come along for the few cases where a child did not want to leave the room, etc. Nevertheless, there were undoubtedly factors which varied from child to child.

Controlling for the interviewer is one of the biggest difficulties in a study such as this. One might choose to have one interviewer for all children so that results across children are more comparable. However, this one interviewer might unconsciously or inconsistently vary his behavior in such a way that the results may be biased. Training several interviewers requires the use of a standard interview protocol, and thus increases the objectivity and reproducibility of the interview. However, the use of several interviewers introduces the possibility that the ideosyncratic behaviors of different interviewers will produce varying results. One might assess the effect of different interviewers by comparing data collected by them from the same sample of subjects. This procedure was not practical in the present study due to both experimental- (learning from the first interview) and subject-related (boredom from doing the same task again, different internal state) constraints. The compromise made in this study was to train several interviewers and to assign them to at most one-half of any cell. In this way any interviewer-generated biases were distributed throughout our sample.

Another interviewer problem concerned the use of new or experienced interviewers for the follow-up interviews. Using the initial interviewers, who were by the second interview quite experienced, could have produced superior results, making the first and second interview data uncomparable. Therefore, new interviewers were trained for the follow-up portion of this study.

A final interviewer problem concerned the need for a blind-experimenter: that is, one who does not know the experimental hypotheses. Anticipation of certain findings could bias interviewer behavior towards different subjects. However, the authors, who obviously were informed interviewers, felt it necessary to have some personal contact with the Ss in order not to lose unanticipated results which might otherwise be ignored by uninformed interviewers. A compromise was made in which the authors did two-thirds of the interviewing in the initial interview, but only one-fifth in the follow-up. Blind interviewers were used in the follow-up interview. The great improvement displayed in the second interview demonstrates that the authors did not bias the initial results.

Implications

Our four main results are briefly restated below.

1) Haptic perception of the geometric shapes used in this study chronologically preceded the ability to produce the shapes by both construction and drawing. The smallness of the shapes used seems to account for the success on the haptic task.

2) Production of shapes is not an all-or-nothing ability. There seems to exist a continuum along an abstractness dimension which predicts that production of a shape using concrete materials (our construction task) is less difficult than one requiring the production of abstract mental images (drawing). For all shapes, construction ability chronologically preceded drawing ability.

3) The complexity of geometric shapes affects a child's ability to reproduce them. Object properties such as curved lines, straight lines, vertices, oblique lines, and symmetrical halves using these oblique lines combine to predice a developmental pattern of production ability. In this study it was found that the order of circle, square, triangle, and diamond described the chronological order both of perception and production.

4) Background A subjects scored significantly higher on the drawing task than Background B subjects, as well as on the construction of the diamond. Although Background A subjects did score somewhat higher on the other construction tasks and on the initial haptic-visual task, the differences were not significant, possibly due to the ceiling reached on these tasks.

Our subjects were able to make haptic-visual discriminations of geometric shapes much earlier than reported by other researchers. Virtually no mistakes were made on this task after age three and one half, and the Background A subjects reached this ceiling by age two and one-half. This success is due to two factors: the small size of the shapes and the simultaneous presentation of the task object and all choice objects. Our subjects did not have to possess or be able to use voluntarily a systematic method of searching the shape -- all of its features were perceptible at all times because the entire shape could be held in the hand. Also, the smallness of the shape would facilitate the coordination of many separate haptic centrations into an image of the total shape, because successive centrations could be made almost simultaneously.

Our haptic results offer no direct evidence concerning Maccoby's assertion that the perception of young children is holistic; but the authors' observations of the behavior of children doing the haptic task and introspection concerning their own haptic behavior provides some indirect evidence. Many children made immediate identifications, saying, "Triangle" as soon as the shape was put in their hand. No attribute was ever isolated in a verbal description; the few descriptions observed were always like "the pointy one" rather than "the one with points". This lack of verbal attribute isolation suggests that even when an attribute was focused on by the child, the totality or wholeness of the shape was so much in the forefront of the child's perception that the isolated attribute was immediately incorporated into a notion of the whole object.

If a child can construct a shape, we can infer that he can discriminate the parts of that shape. The fact that our subjects could construct shapes before they could draw them supports Maccoby's assertion that discrimination of the parts of a figure is necessary before drawing can be done. But the fact that there was a long time lag between successful construction and successful drawing implies that discrimination of the parts is a necessary but not a sufficient condition for drawing a shape. Our results support Zoltan Dienes's contention that construction precedes analysis (1960). If a child was given the parts, he could construct them into a whole much earlier than he could break down (i.e., analyze) the whole into its parts and coordinate these. In particular, the construction task freed him from having to build the shape sequentially, i.e., after parts of the shape were put together in one way, he could adjust the relations of the earlier placed parts. The differences in table 5 between percentages of Ss who received a 3 (built correctly immediately) and who received a 3 or a 2 (trial and error: eventual correct construction, but moved at least one stick in the process of building) reveals that the ability to build a shape in immediate correct sequential order (a prerequisite for drawing success) lagged behind trial and error success by six months (the square) to a year and a half (the diamond). Thus, we would modify Maccoby's hypothesis to say that at least two prerequisites for drawing of shapes exist:

1) the ability to discriminate the parts of the shape, and

2) the ability to operate on a mental image of a shape so that (a) the parts of the shape can be related in a sequential order, and (b) the part(s) of the shape already drawn on the paper can be coordinated with the mental image of the whole shape that is projected onto the paper.

The Background variable in the present study reflects a combination of factors: type of school attended, socio-economic status, working habits of the mother, and amount of time spent in school. Although significant differences in construction and drawing performance were found, it was not the aim of the authors to encourage further research concerning the exact locus of this effect. We feel that energies should instead be directed toward studies which examine what can be done to reduce differences which do exist -- i.e., toward training studies. The imaginative Soviet training studies reported by Zinchenko (1969) indicate fairly clearly that practice in concrete modeling of figures can result in greatly improved drawings for children aged four and above. Goodson's results (reported in Maccoby, 1968) indicate that training in identifying parts of shapes can also improve drawing.

While we advocate research into training procedures, we feel two points must be kept in mind with respect to training of perceptual and representational skills. This training should be training in the general improvement of perceptual and representational abilities, not a narrow focus on the ability to construct or draw a few limited shapes. Knowing how to build a square or a triangle is not important in itself; this ability is only important as an indicator of representational ability in general. Such training can be kept general if it is done with a varied set of materials and for many different shapes. Secondly, such training must be viewed as providing an opportunity for a child to develop whatever capacities he possesses at a given time, rather than as a programmed series of steps that will be successful for any child at any time. As such, the training must consist of creative, imaginative, constructive activities which enable a child to improve his own abilities to discriminate and coordinate parts of figures. Premature or insensitive rote copying of models may lead to a mechanical, chained behavior ("First I do this, then I do this, etc."). This is mere conditioning and will disappear if it is not practiced.

The particular tasks used in our study indicate that the careful choice of task objects can greatly facilitate task success. Researchers and preschool teachers should take careful note of this finding. The size of the materials used here seemed to be the crucial factor both for the haptic identification of the shapes and for the construction with matchsticks. The shapes were small so that children did not have to seek out shape features. In addition, Ss could move quickly from feature to feature to coordinate them into a perception of the whole. However, objects probably get "too small" when they can no longer be handled easily by a child. Motivational components of a task are important too. Schultz (this volume) found that very large shapes (60 cm) used for transformation tasks with elementary school children produced slightly higher performance than small shapes (6 cm). The crucial factor here seemed to be that the very large size of the shapes was intriguing to the children, and they worked longer on tasks involving these shapes. It may be that preschool tasks should involve either small, child-size objects to facilitate cognitive solution of the task or quite large objects which increase action and interest and consequently the amount of time and energy expended on the task. The point to be made here is that size is but one of many variables which may be subsumed under the notion that providing the materials most suitable to a child's given state of development or readiness will facilitate task success. Careful observation and imagination on the part of the teacher is to be encouraged towards this end.

We do not think that the ability to do the tasks in this study are important in themselves and do not propose that they be included in a preschool curriculum. The important implications of the study for preschool education were outlined above. However, this study does reflect one of the most important aspects of research in cognitive development -- that such research leads to the development or refinement of theoretical constructs. Therefore we look to research, such as ours, which bears upon the general theory of spatial development of Piaget and upon specific hypotheses concerning ability and concept acquisition, such as Maccoby's, to contribute to the evolution of theoretical principles which can guide the education of young children.

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The Role of Motor Activity in Young Children's Understanding of Spatial Concepts

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Within the last 20 years certain theoreticians have hypothesized a motor period of ontogenesis during which bodily action is the critical component in the child's developing ability to store information about representational space (Kershner, 1970). As a result, educational programs have been developed which emphasize children's motor activities in their acquisition of spatial notions. Yet, how much justification really exists for a "learning through movement" or a "remediation through movement" approach? More basically, what is the relationship between the realms of motor and intellectual functioning?

In the research reported here, the concept of distance was used to investigate the relationship between motor activity and intellectual functioning. It was hypothesized that a child who, with his own body, journeyed from point A to point B and returned would not necessarily, as a result of the action, comprehend the equality of distance traveled unless he was at the stage of concrete operations.

Related Literature

Movement theorists (Cratty, 1970) who emphasize the role of motor activities for the acquistion of spatial concepts apparently base their arguments upon three sources: "...(a) observations that infants' motor development to a large degree seems to precede their ability to think and to speak, (b) correlative and factorial studies in which scores on intelligence tests have been contrasted to various indices of motor aptitude, and (c) from experimental investigations which have hypothesized that change in intelligence may be elicited by following various programs of physical activity." Delacato (1963), and Godfrey and Kephart (1969) are among those representing this view. Delacato's theory centers around the concept that a defective nervous system can be transformed through the utilization of various patterned activities such as creeping and crawling. He claims that these activities also can aid the child in developing more fully in such areas as reading and visual organization of simple and complex spatial relationships. Cratty (1970) reminds us, however, that within recent years the theories of Delacato have been shown to be generally unfounded, and that Delacato appears unwilling to subject his methodologies to well-controlled experimental procedures.

Godfrey and Kephart (1969) view motor learning as the basis of cognition, and note that the more complex activities such as perception, symbolic manipulation, and concept formation depend upon, and utilize in their acquisition, the more basic motor learning which had its origins in early infancy. They claim that when the child's early motor learning is deficient, more complex learning will be impeded and retarded. Consequently, they state, motor activities are important not only for their own sake, but for the contribution which they must make to the more complex activities the child will be required to perform at later stages (Godfrey & Kephart, 1969).

In drawing conclusions based on studies investigating motor learning, it is important to keep in mind distinctions between gross and fine motor abilities. Wolff and Wolff (1972) comment that these two motor dimensions may be the output counterparts of two functionally independent perceptual systems. Although both systems have motor-perceptual components, fine motor abilities appear to be cortical and to be related to focal or figural perception, whereas gross motor abilities appear to be subcortical and related to spatial perception. Gross motor activities involve locomotion of the entire body in space, whereas fine motor activities require movements predominantly of the fingers and hands in relationship to finely articulated visual stimuli. Clinical observations suggest that many atypical children have serious problems in spatial conceptualization and organization which are reflected in gross and fine perceptual-motor difficulties (Bender, 1938; DeHirsch, 1957; Kephart, 1960; Keogh, 1969). This suggests that impaired motor abilities may underlie spatial confusion, but it may possibly be the reverse. For example, the fields of neurology, language pathology, and mental retardation give some evidence that motor abilities may be influenced by perceptual and cognitive factors (Critchley, 1970; Goldstein, 1948; Goodglass & Kaplan, 1963; Duffy, Duffy, & Pearson, 1975). Consequently, although it is obvious that motor actions are related in some way to a child's understanding of space, the dynamics of this relationship remain largely unknown. Nonetheless, movement theorists recommend the use of both gross and fine motor activities to develop and enrich a wide range of perceptual and cognitive functions. Frostig and Maslow (1970, p.85) suggest the use of movement activities emphasizing such words as "around," "sideways," "before," "after," "in-between," "faster," "slower," "higher," "lower," "wider," and "smaller" to provide children with "...a vocabulary denoting time and space relationship, as well as an understanding of the concepts represented by these words." Frostig and Maslow also consider movement to be critically important in the training of such academic skills as arithmetic and reading (Chapter 8). They comment that movement activities can be used to teach all four arithmetic processes including fractions, noting further that children can learn to compare sizes and distances when they are told, "Let's run to the gate--it's *farther* than the tree," or "Take the *longer* rope" (p. 115).

Most movement theorist particularly emphasize the usefulness of movement to aid the young child in developing *spatial* perception and *spatial* concepts (Barlin & Barlin, 1971; Barsch, 1967, 1968; Hawkins, 1964; Dimondstein, 1970; Russell, 1965). For example, Godfrey and Kephart (1969) recommend that balance activities which relate to gravity are the point of origin for spatial coordination, and note that locomotion is the tool for the use of the total body to explore space. In addition, they view the throwing and catching activities involved in receipt and propulsion as being especially helpful in aiding the child's investigation of movement in space.

Many movement theorists cite the work of Piaget as having been most influential in the development of their own research and theories (Kershner, 1970; Kephart, 1971; Frostig & Maslow, 1973). These theorists frequently comment on the centrality of sensory-motor functions in the development of intelligence, and further propose that it is only by way of direct interaction with the environment through movement that the child can learn about his spatial world. However, they fail to note Piaget's emphasis on actions which are both covert as well as overt, and in so doing misinterpret the term "actions" as referring only to actual motor acts. Nowhere does Piaget state that motor activity constitutes the totality from which the multiplicity of human cognitive abilities stem; yet some of those who desire to set up and implement movement education programs appear to assume that this is the case (North, 1973; Frostig, 1970; Godfrey & Kephart, 1969).

For Piaget, the term "actions" includes the lightning-quick, highly organized systems of internal operations characteristic of adult logical thought, as well as the overt, slow-paced sensory-motor actions of the infant (Flavell, 1963, p. 82). A critical point to emphasize here is that although the neonate's actions are relatively overt and most closely resemble what could be termed motor activity, the internalization of actions begins very early in life. Although the operational level (concrete operations) is not reached until six or seven years of age, the child begins the process of internalization while still an infant. Adult logical operations are then viewed as sensory-motor actions which have undergone a succession of transformations, rather than as a totally different species of behavior. Both involve actions as the common denominator: overt actions in the case of simple schemes; internalized actions in the case of operations (Flavell, 1963, p. 83). Clearly, Piaget's use of the term "actions" refers to far more than simple bodily movements.

In addition to distorting Piaget's use of the word "actions", "learning through movement" proponents also fail to distinguish between two different kinds of activities: those in which there is an isomorphism between the structure of the task and the structure of the spatial concept and those in which there is not. According to Piaget (1956), tasks involving spatial concepts require children to make judgments using systems of relationships that must be gradually coordinated. So, "the problem of learning is essentially how to find a kind of 'best fit' between the structure of a task and the structure of a person's thinking" (Dienes, 1960). Unfortunately, however, for many of the gross motor activities that "learning through movement" proponents suggest, there is no relationship between the structure of the tasks and the structure of the ideas that are presumably being taught. For example, hopping or skipping from one point to another may do little to facilitate a child's acquisition of the concept of distance. In fact, because gross motor tasks often involve a number of striking attributes that bear no relationship to the concept being taught, they can actually confuse children who have already learned to use the concept in simpler situations.

When movement proponents state that a young child can best develop spatial concepts through active utilization of the total body moving in space, they appear to be unaware of Piaget's frequent mention of preoperational children's lack of fixed notions of space and their many confusions of spatial qualities with temporal and effort and movement qualities. They also ignore the fact that young children are so often fixated on end points or goals that "going" is often viewed as farther than "coming back" in journeys of equal distance (Piaget, 1946; Piaget, Inhelder, & Szeminska, 1960). Consequently, tasks in which children are motorically active can actually be confusing because they draw the child's attention to characteristics that are irrelevant to the concept being taught. For example, if a child is supposed to learn about "symmetry of distance," a task in which the child actually walks a line A_____B in one direction and then in the opposite direction, it may actually be more difficult than a task where he was less active and merely moved an object along a path in both directions, or in tasks in which he was even less motorically active and merely watched the experimenter do so. All three of these tasks are conceptual, not some motoric and others cognitive. However, it is probable that as the child develops, there will be some décalage, with the child judging correctly on one task before another. Some tasks may involve either fewer total variables to process, or fewer of the kinds of variables which act to confuse young children. These "simple" tasks may, therefore, be easier for children in the early stages of the development than are other seemingly equivalent tasks dealing with the same spatial concept.

If children cannot comprehend the equality of distances as readily on some tasks as on others, then the next question is, "Which aspects of the situation might confuse them: effort, e.g., 'I went farther when I carried the basket because it was so hard for me,' goalness, e.g., 'I went farther going 'cause it's always farther to go somewhere than to come back,' time/speed, e.g., 'I went farther when I carried the basket because it took me longer,' or perhaps others, e.g., 'I went farther with the groceries because I took more steps?"

The following then were the hypotheses of the study:

(1) There will be no task décalage between gross motor versus fine motor, or observer versus participant tasks in the ability to comprehend and conserve the equality of symmetrical distances.

(2) However, within the gross motor realm, performance on walking tasks will be superior to performance on run-jump tasks because the walking tasks embody fewer time/effort/movement variables to confuse preoperational children.

(3) Children who are not yet operational regarding the concept of symmetry of distance will judge one distance as farther than the other ($AB \neq BA$). Furthermore, on those tasks in which effort is a manipulated variable, the direction involving greater effort (weight or skill) will be judged as "farther."

Method

The subjects for this investigation were 142 predominantly white upper middle class suburban children, ages 44-108 months (see Table 1). These subjects were asked to judge and justify the equality of a distance traversed back and forth either (a) by themselves (gross-motor action), or (b) by a doll moved (1) by them, or (2) by the investigator (fine-motor action as manipulator or as observer). Eleven tasks were constructed which varied in terms of gross-motor (whole body) versus fine-motor (manipulation of a doll) action. Within gross-motor they varied as to (a) amount of effort in carrying a load, and (b) amount of skill (walking versus running/jumping); and within fine-motor varied in terms of (c) observing the investigator move a doll versus move the doll oneself, and (d) position of the doll in relation to the subject (parallel versus perpendicular). Figure 1 illustrates the construction of the eleven tasks.

A screening test was individually administered to each child in order to eliminate at the outset those children who were unable to demonstrate an understanding of the term "farther" even under the most clear-cut situational conditions.

The study utilized a repeated measures design whereby each of the 142 subjects did all 11 tasks. The order of presentation of the tasks was varied among the participants.

Table 1

Subjects, Median Age,* Range, and Sex

Group	Median Age (months)	Range (months)	Male	Female	n
1	50	44-53	4	12	16
2	57	53-59	11	5	16
3	63	60-65	2	14	16
4	69	65-71	6	10	16
5	74	71-76	10	6	16
6	80	76-83	11	5	16
7	87	83-91	7	9	16
8	95	92-101	10	5	15
9	105	103-108	8	7	15
			69	73	142

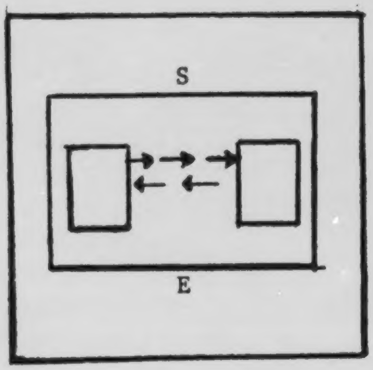
* All ages rounded upward at the 15th of the month and above.

Figure 1. Eleven test tasks

On floor with two sheets of paper as border

Task 1 - E moves doll parallel to S

Task 4 - S moves doll parallel to S



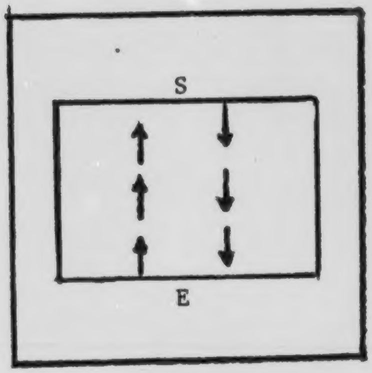
E and S sit on floor facing each other

Task 2 - E moves doll perpendicular to S beginning at S

Task 3 - E moves doll perpendicular to S beginning at E

Task 5 - S moves doll perpendicular to S beginning at S

Task 6 - S moves doll perpendicular to S beginning at E



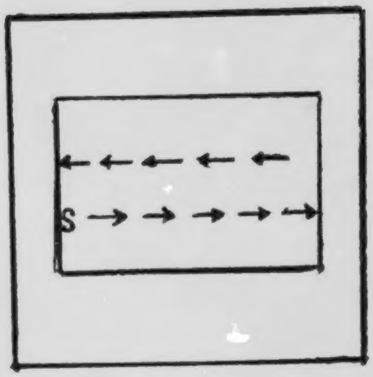
S walks from wall-to-wall in room

Task 8 - S carries basket to one wall, leaves basket and returns empty-handed

Task 9 - S walks empty-handed to wall, picks up basket and returns with load

Task 10 - S runs to wall, jumps return journey

Task 11 - S jumps to wall, runs return journey



Test Procedure

Task 1 - The Experimenter Moves Doll Parallel to Child

Sitting on the floor approximately 0.5 m apart and facing the child, the experimenter placed two sheets of blank paper about 30 cm apart on either side of the child and said, "This doll is going to take a walk. See, she starts here (at the edge of one sheet) and walks right to here (the edge of the other sheet). Now, she turns around and walks back to right where she started." (The starting line alternated between children.) The experimenter then executed the doll's movements appropriately and asked, "Did the doll walk (or go) the same distance when she went from here to here (points) as she did when she went from here to here, or did she go farther one time than she did the other?"* With older children, the question was simplified: "Did the doll walk the same distance both times, or did she walk farther one time than the other?" If the child just said "farther," the experimenter probed, "Which way?" The experimenter avoided using the terms "coming" or "going," as it was noted during a preliminary study that young children tend to confuse these terms, and often use one when they mean the other. However, a child will frequently spontaneously say, "She went farther going there," or "She went farther when she came back."

The experimenter wrote the child's response (Judgment Response) verbatim on the test protocol, refraining at all times from evaluative statements such as, "That's right," or "That's wrong." Next she asked, "How do you know it is the same distance (or farther)?" and wrote down the child's justification (Justification Response). If a child was unable to give reasons for his/her judgments or gave responses such as, "I just know 'cause I'm real smart! " the experimenter attempted to elicit further justification if possible. Thus, for each of the 11 tasks, the protocol contained two kinds of responses; judgments and justifications.

Tasks 2-6 utilized the same procedure, varying only in terms of (1) the child's observing the doll's movement versus moving the doll himself, (2) the doll's movement as perpendicular to the child versus parallel to him, and (3) the doll beginning its journey at the child versus at the experimenter.

Tasks 7-11 used the same basic structure, but on the gross as opposed to the fine motor level.

Task 7 - Child Walking Empty-Handed

Beginning at one of the walls furthest apart (about 5-6 meters; the starting wall alternating between the subjects), the experimenter said, "Stand at the wall, making sure your heels touch the wall. Walk to the other wall until your toes touch the wall. Then, turn around and walk right back to where you started, and make sure your toes touch the wall."[*] After the child returned, the experimenter then asked, "Did you walk farther when you went from here to here (points), or did you walk farther from here to here (points), or did you walk the same distance both times?"

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* In all of the tasks, the questions were alternately asked with the "farther" clause first.

Tasks 8-11 utilized the same procedure, varying only in terms of (1) walking while carrying a basket on one journey--either (a) outward, or (b) return, or (2) the child's running and jumping versus walking the distance--either (a) run outward - jump return, or (b) jump outward - run return. Questioning and writing of the judgment and justification responses followed the standard form. (See Musick, 1976, for further details of test procedure.)

Analysis

It was noted during the testing that some younger subjects tended to give correct judgment responses, yet when asked to justify this response, would change their minds, or say, "I don't know. I just guessed." That is, not all responses seemed equally acceptable. Therefore, the individual protocols of judgment responses were scored in part on the basis of the type of justification offered for the judgments of equality of distance. A score of 1 was assigned for a correct judgment ("same distance") which was also adequately justified, and a score of 0 was given for either of three classes of judgments: (a) vacillation, (b) an incorrect judgment ("farther"), or (c) a correct judgment inadequately justified. This particular scoring procedure was chosen because the correct but inadequately justified response clearly indicated subjects lack of understanding of the concept involved. Typical responses in this category either referred to "magical" reasons such as "God told me," or focused on irrelevant factors such as, "Both sides have no door in them." The results would have been less accurate had the scoring been "easier" as there was no indication that subjects categorically really understood the concept but were too linguistically immature to justify it adequately. There were children who responded correctly but gave linguistically immature justifications such as, "Well you know, it's all the same space, this space doesn't change, no matter what happens here." These responses were scored as adequate because the child, though lacking language facility, did display his understanding of the concept, whereas the other subjects did not.

The highest possible overall score any individual subject could obtain was 11 (scores of 1 on 11 tasks). Group scores could not exceed 16 (or 15 in the case of groups 8 and 9, n = 15) on any one of the 11 tasks. The maximum possible total group score for the entire test was therefore 16 x 11 = 176 for each of groups 1 - 7 (n = 16) or 15 x 11 = 165 for each of groups 8 and 9 (n = 15).

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[*] The experimenter's insistence on the child "touching" or going "right back" or "exactly to where you started" stemmed from noting (during a preliminary study) that children of 5-6 years are extremely literal and would often say, "I went farther going there 'cause when I came back, I stopped right here (pointing to a spot 1-2 cm from the wall).

Several types of justification responses were extracted through coding of all protocols. The four classes of judgment responses (3 classes of incorrect and 1 class of correct) are described in Table 3.

Table 3

Types of Justification

Score of 0: Example

(A) No definite response and no justification

1. Vacillation: "It's the same--no, it's farther going, no, it's farther coming back."

(B) Incorrect judgment response

2. No justification: "It's farther to go there (points)." "How do you know that?" "I just know it, I just see it."
3. Spatial confusions: "It's farther to go 'cause that side of the room is longer--it's bigger there."
4. Effort confusions: "It's harder to carry the basket home 'cause it's so heavy and hard for me."
5. Goal-Related confusions: "It's always farther to go someplace than to come back."
6. Egocentric confusions: "It's more far for the dolly to go from me to you than to come back to me 'cause she was farther away from me--see?"
7. Time/Speed confusions: "It went farther when I ran 'cause it's faster than jumping."
8. Additivity confusions: "It's the coming back time that's farther 'cause you went there, and then came back--you've gone two times!"

(C) Correct judgment responses but with incorrect or inadequate justifications

9. Unable or unwilling to justify: "I just know in my head it's the same." "But how do you know?" "The room told me, I guess."
10. Incorrect justification: "It's the same both ways 'cause there's a window on both sides of the room."

Score of 1: Correct judgment responses with adequate justification

11. Action-Oriented: "The doll went here and then just turned around and went back where she started (based on child's or doll's actions)."

12. Space as stable, unchanging, whole: "See, it doesn't matter what you do, the room stays the same size. It's all the same space."

13. Early Euclidean-spontaneous measurement: "I could just walk back and forth and count my steps--it would be the same number both ways."

Percentages of correct judgment responses at each age level (9 groups) were computed for each of the 11 tasks individually, and then combined overall. From these perçentages, a mean percentage score for each age group and for each of the 11 tasks was obtained. Next, a linear trend analysis was performed for all 11 tasks combined. In addition, the age of passing each of the 11 tasks (separately and combined overall) was computed using 75% correct as the criterion for passing.

A 9 x 2 (age x task) analysis of variance with repeated measures on the second factor (i.e., task) was performed for comparisons 1-11 noted below. The term "tasks" here refers to the pairs of comparisons made: e.g., Gross Motor Tasks 1-6 versus Fine Motor Tasks 7-11, or Basket Going (task 8) versus Basket Coming (task 9).

The following comparisons of judgment performance were made on the data:

1. All fine motor versus all gross motor (tasks 1-6 versus tasks 7-11).
2. Observer versus participant (tasks 1-3 versus tasks 4-6).
3. Perpendicular versus parallel movement (tasks 2,3,5,6 versus tasks 1,4).
4. Walking empty-handed versus carrying basket (task 7 versus tasks 8,9).
5. Walking only versus running-jumping (task 7 versus tasks 10,11).
6. Carrying basket versus running-jumping (tasks 8,9 versus tasks 10,11).
7. Basket going versus basket returning (task 8 versus task 9).
8. Running-jumping versus jumping-running (task 10 versus task 11).
9. Walking in general versus running-jumping (tasks 7-9 versus tasks 10,11).
10. Lesser versus greater effort in gross motor (task 7 versus tasks 8-11).
11. Doll beginning at child versus doll beginning at experimenter (tasks 2,5, versus tasks 3,6).

The ANOVAs were followed by Neuman-Keuls Tests for differences between pairs of means. This post hoc measure established and defined more clearly the boundaries of sequential developmental stages in the acquisition of the concept of symmetry of distance (See Musick, 1976 for details).

A chi square analysis was performed on the "farther going" versus "farther coming back" errors on those gross motor tasks where effort was a manipulated variable (tasks 8, 9, 10, 11). This was to determine whether S's errors on these tasks were a result of confusions due to qualities, or other factors such as time/speed or goal-related confusions.

Results

The results presented in Table 4 indicate that younger children were unable to understand the symmetrical character of distance AB = BA. Not until the age of approximately 6 years, 5 months (77 months) were subjects able to comprehend that a path traversed in one direction was the same distance as its return journey. No prediction had been made as to the direction of the errors; whether the child would judge $AB > BA$ or the reverse, $BA > AB$. An analysis of all incorrect responses revealed only a 25 point difference between the number of "farther going"(236) versus "farther coming back" (211) errors. The linear trend analysis confirmed the existance of an upward linear trend, F (1,135) = 341.99, $p < .001$.

As hypothesized, there was no significant difference between performance in fine versus gross motor, or observer versus participant tasks in the ability to comprehend and conserve the equality of symmetrical distances (see Table 4). Subjects who were unable to comprehend the equality of the distance AB = BA on one form of task were generally unable to do so on the other. Conversely, subjects who were able to comprehend the symmetry of the distance on one form of task generally were able to do so on the other.

In particular, the absence of differences in performance between observer and participant on fine motor tasks gives additional support to the author's view that a child need not "act upon" space physically in order to comprehend its properties.

Also, as hypothesized, jugment performance following the walking tasks was significantly superior to that following running and jumping tasks. The comparison of all the tasks which involved the subject in walking (tasks 7, 8, 9) to the more difficult gross motor tasks requiring a subject to run and jump the two equal distances (tasks 10,11) to determine if the effort/time/movement factors in the run-jump tasks would affect performance revealed significant main effects for both age (F, (8, 135) = 32.99, $p < .001$) and task (F (8, 135) = 10.92, $p < .001$).

These results are similar to those in which walking-only was compared to running and jumping (F (8,135) = 27.85, $p < .001$ (age), and F (8,135) = 15.42, $p < .001$ (task)), and to those in which carrying baskets (which is essentially a walking task) is compared to run-jump tasks (F (8, 135) = 33.79, $p < .001$ (age), and F (8, 135) = 7.14, $p < .01$ (task)). No significant differences

Table 4
Percentage of Correct Answers* by Age & Task

Age-Group	Group Median Age (Months)	n	Fine motor Observer			Fine motor Participant			Gross motor					Mean Percentage Score
			1	2	3	4	5	6	7	8	9	10	11	
9	105	15	100%	100%	100%	100%	100%	100%	100%	100%	100%	93%	93%	99%
8	95	15	100	100	86	100	93	86	93	100	100	86	86	94
7	87	16	100	100	93	100	100	93	100	93	93	87	87	95
6	80	16	93	87	93	87	87	93	87	93	81	75	75	86
5	74	16	75	68	68	75	75	81	68	68	68	56	56	69
4	69	16	56	50	31	56	56	50	75	62	56	43	43	53
3	63	16	12	6	6	0	0	6	18	18	0	6	12	8
2	57	16	12	6	0	6	6	6	12	0	12	6	6	7
1	50	16	6	6	6	12	12	1	0	0	0	0	0	4
	Mean Score:		61.6	58.1	53.7	59.6	58.8	57.9	61.4	59.3	56.7	50.2	50.9	

Mean Age at Passing =74 77 77 74 74 73 69 75 77 80 80
Mean Age at Passing for all 11 tasks at 75% correct = 77 months. (6 years, 5 months)
N = 142

*Correct answer is judgment of "same distance" which is also adequately justified. Mean scores for significant differences can be found in Musick (1976).

Table 5
Comparisons of Judgments Performance
Result of Statistical Analysis[*]

COMPARISONS	F
1) All fine motor versus all gross motor (Tasks 1-6 vs. Tasks 7-11)	1.62
2) Observor versus participant (Tasks 1-3 vs. Tasks 4-6)	.73
3) Perpendicular versus parallel movement (Tasks 2,3,5,6 vs. Tasks 1,4)	3.54
4) Walking empty-handed versus carrying a basket (Task 7 vs. Tasks 8,9)	2.83
5) Walking only versus running-jumping (Task 7 vs. Tasks 10,11)	15.42***
6) Carrying basket versus running-jumping (Tasks 8,9 vs. Tasks 10,11)	7.14**
7) Basket going versus basket returning (Task 8 vs. Task 9)	2.18
8) Running-jumping versus jumping- running (Task 10 vs. Task 11)	1.00
9) Walking in general versus running-jumping (Tasks 7-9 vs. Tasks 10,11)	10.92***
10) Lesser versus greater effort in gross motor (Task 7 vs. Tasks 8-11)	12.59***
11) Doll beginning at child versus at experimental (Tasks 2,5, vs. Tasks 3,6)	2.46

** p .01
*** p .001

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[*] Results presented in F column are main effects for Task only. All comparisons revealed significant main effects for age $p < .001$. No significant interactions were revealed by the analyses.

were found between performance on the walking only versus walking with basket task. It appears, then, that children are better able to judge the equality of distances which they have walked (even while carrying a heavy basket) than of distances which they have run and jumped. This issue will be discussed further in the Discussion section. Table 5 illustrates the results of the statistical analyses computed for the 11 comparisons.

The third hypothesis that preoperational children would confuse the spatial notion of distance with effort qualities was only partially supported. Although subjects were found to follow this pattern on tasks where the effort quality involved was weight (basket tasks 8-9), the reverse pattern was found on tasks where the effort quality was motoric skill (run/jump tasks). On these tasks, a significantly greater number of subjects judged as farther the route on which they ran, although running requires less muscular effort (and skill) than does jumping, especially for young children.

An attempt to clarify this apparent contradiction (x^2 (1 df) = 14.90, $p < .01$ (1 df) (see Table 6) began with an analysis of both type 4 (effort) and type 7 (speed/time) justification responses. There were only 14 justifications based on effort qualities; of these, 12 were in the basket tasks (tasks 8 and 9). Upon further questioning, many of these subjects would add, "Well you know, 'cause it's heavy (or hard) so it takes longer."

Table 6

Route Judged Incorrectly as "Farther"

Task	Greater Effort	Lesser Effort
8	34	14
9	24	19
10	17	32
11	19	35
Totals	94	100

Therefore, even on those tasks where the effort qualities of the situation appear to confuse children, temporal qualities may be involved to some degree.

A further analysis of justifications for incorrect judgments on the basket tasks reveals that although subjects more frequently judged the route in which they carried the basket as farthest, when giving justifications for these judgments no one type of response was predominant. The carrying of the basket was often not even mentioned, nor stated to be the cause of the symmetry. The impression given is that the subjects were *distracted* by the effort involved in carrying the basket and so misjudged the distance. However, when questioned about their judgments, they were not really certain *why* it seemed farther so they justified their decision on the basis of whatever quality or characteristic of the total situation was most *salient* for them at the moment of questioning. Typical examples include:

> Type 3 (spatial confusion): S - "When I carried the basket it made that path longer." E - "Do you mean it stretched it out?" S - "Yes, it pulls it out longer."
>
> Type 4 (effort confusions): S - "If it's hard to carry something, that way is the longest."
>
> Type 6 (egocentric confusions): S - "Grandma's house is more far from me."
>
> Type 8 (additivity confusions): S - "Well, I went to the store and now I'm back home, so this way is the farthest."

The "distraction" caused by the effort factor in the basket tasks is understandable in view of the inability of preoperational children to focus on more than one dimension at a time and the tendency to center their attention on only one aspect of a situation. It is also in keeping with the egocentric thought of young children to believe that the properties of time and space are in some way related to, or dependent upon, their own actions (Piaget, 1927, 1946).

The responses to the run/jump tasks (10 and 11) provide a striking illustration of the interdependence of the young child's conceptions of movement, time, and space. In Piaget's view, time, space, and movement form a whole at first. Young children are often unable to clearly differentiate among them. For example, if one car moves faster than another, it is viewed as having gone farther, even if both cars stop at the same end point. Or, the reverse may occur where one car is believed to have moved faster than another, merely because it has gone farther, having stopped at a more distant point (Piaget, 1927, 1946).

An analysis of the judgment responses on the run/jump tasks indicates that a significantly greater number of subjects chose the less effort route--running--as farther on both tasks (10 and 11) (see Table 6). An analysis of justification responses reveals that the judgments of running as farther are based primarily on the temporal qualities inherent in the motor acts of running and jumping which appear to be superordinate over any effort factors involved. For all tasks combined, there was a total of 53 type 7 (speed/time confusion) responses; 87% (46) of which occurred on tasks 10 and 11. When subjects judged the running route as farther, the time/speed justifications were most frequent: "Running makes it farther 'cause it's faster." "It's always farther when you run 'cause you get there more quick than when you jump." "Runs are always more far than jumpers, 'cause jumpers is slower and they take a longer time." Even those subjects who judge the jumping (greater effort) route as farther do so almost exclusively on the basis of temporal as opposed to effort qualities. "Jumping takes a longer time, so you go farther." "Jumping is real slow, so you've gone farther to get there than when you run."

Here we see the preoperational child's tendency to center on only one dimension of a situation as he focuses attention on only the time/speed factors, and is therefore unable to keep purely spatial factors in mind. In addition, we see the irreversibility of notions that cannot deal with space which has been "broken" by running and jumping, by making it back into a whole, or by noting that one route is just the reverse of the other.

Conclusions

Although no significant overall differences were found between performance on fine motor and gross motor tasks, certain individual tasks were somewhat more difficult than others. For example, statistical analysis revealed a significant difference in performance between the walking tasks (even those on which a basket is carried) and the run/jump tasks. This result illustrates the subtle interdependence of the motoric and cognitive aspects of full body movement in space, and indicates that to view all gross motor activity as falling within a single category is grossly misleading. Not only were young children not any better able to understand and conserve the symmetrical nature of distance traversed when they actually moved this distance themselves, but under certain gross motor conditions they were even less able to do so! Far from acting to facilitate spatial understanding, certain gross motor tasks appear to impede it. Although most children's notions of distance remained the same under both the fine motor and the walking tasks, this was not the case in relation to the run/jump tasks. Indeed, there were many young children who comprehended and conserved the equality of the distances on the walking and fine motor tasks, but who appeared to "lose" or lack this understanding on the two run/jump tasks. For example, one subject said: "It's the same distance when the doll walks it, and when I walk it, but it's not the same distance to run it or jump it." _E_ - "Why is that, Josh?" _S_ - "Well, when a person runs, it's just longer that way." _E_ - "But you just told me that walls don't move, so how come you're saying it's farther when you run?" _S_ - "It just makes it kind of stretchy...Oh, I don't really know why. It's just farther!" This kind of justification was not uncommon and is illustrative of a point made by Meyer. _She notes that comprehension of spatial relations occurs on different levels of mental evolution, according to the difficulties of the problem_. Like Piaget, she finds that it develops by a continuous genetic process of which developmental stages mark the fundamental steps. Whenever the child's mind "...encounters problems too difficult to solve with the functional instruments it has at its disposal, the process of comprehension reverts to the same mechanisms long left behind on other levels" (Mayer, 1940, p. 149).

When there was an overload on the temporal or effort factors involved in running and jumping tasks, subjects appeared to regress temporarily to earlier, more egocentric modes of thought. A striking illustration of this "regression" was given by those subjects who were presented with the reverse order of tasks; gross motor first, fine motor second. In many of these cases, subjects would state their judgment of the asymmetry of the distance of the run/jump tasks, but when presented with all the other (gross and fine motor) tasks, would say, "_Now_ it's the same distance 'cause it's walking." Or, "I can see it's the same both ways when the dolly walks, or when I walk 'cause we don't go so fast, and we don't take so many steps then."

Young children are so "caught-up" or "captured" within this kind of activity that they are unable to extract themselves; to step back and view the system with the subjectivity necessary to understand the relations within it. This may be accomplished more easily when the child engages in simple fine motor activity (observer or participant) such as that utilized in this study, or in the simpler gross motor activity of walking which is not as heavily saturated with time, effort, speed, and movement qualities as are the more complex activities of running, jumping, skipping, sliding, and hopping so often used by movement educators to introduce children to spatial concepts.

Since, according to Piaget, the child must coordinate a system of relations in order to grasp distance concepts, he must be able to see the entire system. The fewer distractions inherent in the task, the more quickly or easily the child will be able to focus on the relevant variables and therefore comprehend the underlying structure of the system.

Spatial concepts are not immutable forms constructed at one point during the growth of intelligence and then immediately applicable to all situations. On the contrary, they appear to be correlates of activity at various levels (Piaget & Inhelder, 1956). These concepts are, as Meyer (1940) notes, the instruments which the mind renews and recreates during its activity, and their creation follows similar laws at all levels. The beginnings are found in the earliest or sensori-motor period. They are then reconstructed and reconstructed again on a level of practical activity in the period following sensori-motor development, and then in the operational stage.

The results of the investigation indicate that at the preoperational stage the child is particularly susceptible to confusions arising from a configuration of distracting factors surrounding certain gross motor acts. That is, the boundaries between notions of space, time, movement, and in certain instances, effort, still appear somewhat unclear. The results of this investigation indicate that the temporal, spatial, movement, and effort qualities inherent in certain motor acts may, at times, interfere with the child's comprehension of spatial relations causing the child to misjudge and to attribute change to a property he had previously viewed as invariant. Therefore, a child's ability to "know by doing" is sometimes limited.

In infancy, the child did physically explore space. However, as the child begins to move away from reliance on the sensori-motor aspects of spatial awareness towards the reversible objective space of operational intelligence, he passes through a preoperational stage during which he is particularly susceptible to confusions arising from certain factors inherent in motoric activity. The more the motoric activity involves these factors (time, effort, etc.), the greater the possibility of spatial confusion. Simple gross motor acts, such as walking, would be at one end of the continuum which moves up through running and jumping and beyond to such complex motor acts as skipping, sliding, hopping, and leaping.

A logical analysis should be undertaken of any activity which attempts to introduce a child to a system of operations which he must coordinate. To promote this coordination we should look for those kinds of activities which have some relation to that system. The more closely related, or isomorphic the structure of the concept and the structure of the task, the more easily it will be grasped. Even those tasks in which there is an isomorphism often contain two potential categories of activities: gross motor and fine motor. Fine motor tasks and simple gross motor tasks may indeed be better for younger children; yet, the fact that a task is fine or gross motor may not be of primary importance. Rather one should look at the similarity between the structure of the task and that of the system or concept.

In general, any but the simplest gross motor tasks appear to introduce (as would highly complex fine motor tasks) too many extraneous factors into the system, which act to distract the child and impede his ability to grasp the concept and its underlying structure.

A final comment deals with motoric activity and older children. The older subjects participating in this study did appear to enjoy and to more fully involve themselves in the gross motor tasks than in the fine motor tasks. Although there was no difference in performance between the two forms of task, older children who clearly understood the symmetry of the distance appeared to derive a "bonus" from all the gross motor tasks. This bonus was obtained in two ways; one motoric, the other conceptual. The first had to do with an apparently genuine pleasure in the use of the body to move through space; a personally experienced and exhilerating exploration of space based on one's own skilled actions, in which movement, time, and space are fused. The second bonus revealed older children's delight in the intellectual exercise involved in separating the extraneous from the relevant variables in the gross motor acts. For example, a child of 7 years, 6 months, told the experimenter, "Even if you go fast, or slow, it doesn't change the space. The space always stays the same." Or, "I could carry 16 baskets and take a real long time, and be so tired, but that can't make the walls move back and forth. Now, if I'd walked a big crooked, curly way, I'd have gone more distance, but I just went straight back and forth both times in this same piece of space!"

The older child who already possesses an operational grasp of spatial relations may subsequently enjoy and learn still more about space through active exploration in which full body movement is utilized. Eventually, he can develop the ability to use his body as the instrument for the creation of movement compositions in space and time.

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Mathematical Foundations for the Development of Spatial Concepts in Children

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Chicago Circle

We live in space. In order to deal effectively with the world around us, we must learn to function in space. Moreover, spatial concepts underly and play an important role in the development of many other concepts. The use of the number line, for example, in representing certain properties of numbers presupposes some specific understanding of space. It is clear that an adequate understanding of space is absolutely essential!

What constitutes an "adequate understanding" in this context? As part of our normal functioning in space, we perform many tasks fully anticipating certain outcomes. In moving a glass of milk from the counter to the table we implicitly accept the fact that the shape (a Euclidean property) will remain unchanged. Indeed, we would be quite surprised (to put it mildly) if the glass were to undergo a topological transformation which drastically changed its shape! Similarly we measure the length of the cabinet in the store and the space along the wall in the dining room, and decide with reasonable confidence whether or not the cabinet will fit into the space. It is this ability to say what will happen if a certain action is carried out, without actually carrying out the action--this ability to anticipate or predict the outcome of certain actions which, in our view, constitutes "adequate understanding." In this respect, then, most of us do acquire a somewhat adequate understanding of space. Yet for most people it is inadequate to deal with the following situation: figure 1 presents us with two views of the same cube; there is a circle in the center of the hidden face. Which faces are opposite which faces?

Figure 1

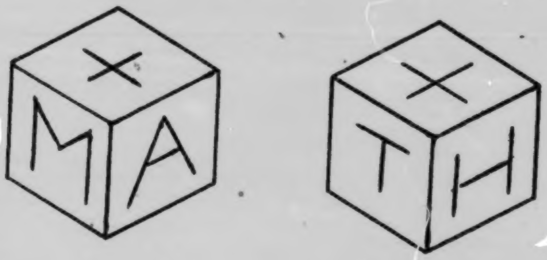
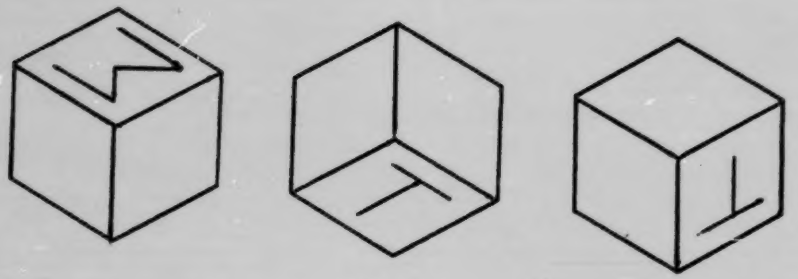


Figure 2 presents three views of this same cube, but incomplete. Complete these views.

Figure 2



The difficulty many people encounter with these tasks serves to point up some of the inadequacies of our spatial concepts.

One objective of mathematics education must surely be to ensure, or foster, or somehow bring about an adequate understanding of space in children. In order to carry out this objective, we must first understand how these concepts develop in children. This question has been studied by a number of investigators, foremost among them is Jean Piaget. His work is frequently cited as the justification for many different and often contradictory theses! In view of the great influence exerted by his work, it is of the utmost importance that this be examined very carefully. One thing is clear. Any investigation of how a particular concept develops must necessarily be based on the investigator's preconceived notion of the nature of that concept. One cannot investigate the development of a concept in children without a thorough understanding, by the investigator, of just what it is all about. In particular, it is important to determine what is the view of space upon which Piaget bases his studies.

While it is difficult to extract this information from the writings of Piaget, a study of his work does provide certain clues. He views cognitive growth in general as the gradual mastery of invariant properties under progressively more complex systems of transformations. With regard to spatial concepts, one of his most frequently cited conclusions is that topological relations are grasped by the child before Euclidean relations. He goes on to assert that whereas historically Euclidean geometry was developed prior to topology,

> ...this psychological order is much closer to modern geometry's order of deductive or axiomatic construction than the historical order of discovery was. It offers another example of the kinship between psychological construction and the logical construction of science itself. (1965, p. 409)

It seems quite clear from this and other writings of Piaget that his views of space and geometry were strongly influenced by what has come to be called Klein's Erlanger Programm.

Felix Klein, on accepting a chair on the faculty at the University of Erlanger, in his inaugural address presented a definition of geometry as the study of properties invariant under a group of transformations. This leads to a simple classification of geometries over a set or a space in terms of these transformation groups. Topology, as the most general geometry is therefore, in some sense, the most basic, with projective geometry as a subgeometry of topology and Euclidean geometry a subgeometry of projective geometry. This, of course, is precisely the "psychological order" of development according to Piaget.

The influence of the Erlanger Programm on Piaget's views of space and the impact of Piaget's views on mathematics education no doubt accounts to some extent for the great interest presently manifested in the approach to Euclidean geometry through transformations--transformation geometry (this still seems to be regarded as the geometry underlying space!).

According to the Klein definition, in terms of the Erlanger Programm, Euclidean geometry (of the plane, say) is the study of the invariants under the group of motions. One important theorem in the Euclidean geometry of the plane is that motions may be classified as follows: There are four basic classes of motions other than the identity--(i) reflections (flips), (ii) translations (slides), (iii) rotations (turns), and (iv) glide reflections. Every motion other than the identity is in one and only one of these classes. This, in turn, has led to a great deal of interest in the question of how children develop an understanding of each of these classes of motions (usually omitting the fourth class for some reason!).

The influential role played by the Erlanger Programm in the work of Piaget and others makes it a necessary prerequisite to an understanding of these works that the Erlanger Programm itself be fully understood, particularly with respect to Euclidean space. What are the various transformation groups and their geometries? What are the invariants of these groups? What is the relation of some of the research to the Erlanger Programm? How does this conception of geometry fit with the child's development of spatial concepts? These are but some of the questions to which we must address ourselves.

The main part of this paper will be devoted to a thorough analysis of the Erlanger Programm in relation to Euclidean space. However, for simplicity of exposition we will deal with the plane. The main section will be subdivided into subsections, each dealing with a geometry of the plane determined by a transformation group. This will include a discussion of how the group is obtained as well as a study of the properties invariant under that group.

A second section will present a brief discussion of some of the research in the light of our analysis.

The third section will briefly discuss the relation between the view of geometry as expressed by Klein and the child's experiences in space. In this section, we introduce our own views of geometry and space and present the case for the greater compatability of this view with the child's experiences in space. We conclude by indicating some implications of this view.

The Klein Erlanger Programm

The Group of Motions

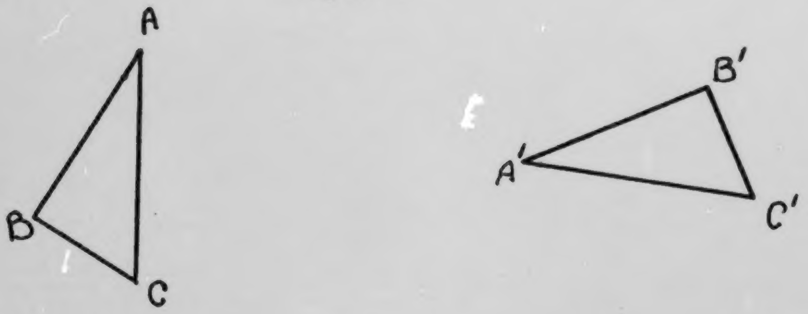
Klein's definition of a geometry as the study of those properties invariant under a group of transformations, like most definitions in mathematics, tells us very little about geometry unless we already know a great deal! In the first place, we must understand the context of the definition; given is a set or space and a group of transformations on the set. A "geometric" property of a figure (or subset of the space) is one which is preserved or left invariant by every transformation of the group. This definition is general enough to include situations we do not normally regard as geometry. If, for example, G is a group and $\mathcal{A}(G)$ the automorphism group of G, then the property of a subset of G of being closed under the binary operation of G is invariant under any automorphism of G and hence a "geometric" property. On the other hand, Klein's definition overlooks or excludes many things in geometry as we usually think of it. Nevertheless, it does provide a convenient framework for classifying geometries. If G is a group on a set S and H is a subgroup of G, then H is also a group of transformations on S, and hence, also determines a geometry (over S). This geometry determined by H is a subgeometry of the one determined by G.

For simplicity of exposition we confine our attention to the Euclidean plane.

In the early part of his Elements, Euclid focused his attention on conditions for the congruence of two figures. It is clear from some of his proofs that he regarded two figures as congruent if one could be superimposed on the other--that is, if one of the figures could be lifted up and fitted onto or made to coincide with the other. Unfortunately, Euclid failed to formulate this concept (and others as well) in precise mathematical terms. Modern developments of geometry, which rectified the various deficiencies in Euclid's Elements, usually sidestep this question by taking "congruence" as an undefined relation. Klein took a different approach by providing the necessary mathematical formulation of Euclid's principle of superposition. This he did in terms of transformations.

Consider two triangles, $\triangle$ ABC and $\triangle$ A'B'C' (figure 3). How can one determine whether or not they are congruent? One way would be to cover $\triangle$ ABC with a clear acetate sheet and trace the triangle. If the acetate can be removed and replaced so that the tracing on the acetate covers or coincides with the triangle $\triangle$ A'B'C', then the two triangles are congruent. We describe this action as superposing the triangle $\triangle$ ABC onto $\triangle$ A'B'C'.

Figure 3



A more precise formulation of this action leads to the notion of a transformation, and, in particular, a motion.

In considering whether △ ABC can be superposed on △ A'B'C', we are not at all interested in what happens to the acetate (after tracing the △ ABC) between the time the acetate is removed from the paper and when it is replaced with the tracing covering the triangle △ A'B'C'. All that concerns us is that in the initial position the tracing on the acetate coincides with, or covers, the triangle △ ABC, and in the final position it coincides with the triangle △ A'B'C'. In the initial position of the acetate, each point on the acetate lies over or covers exactly one point of the paper. The point on the acetate serves as a surrogate for the point on the paper which it covers. In the final position of the acetate, this surrogate point on the acetate covers another (possibly the same) point of the paper. Thus, with reference to the preceding situation, the tracing of the vertex A lies over or covers the vertex A in the initial position and over the vertex A' in the final position. The net result of this is to create a situation wherein, subject to certain physical limitations, to each point of the paper (say A) there is assigned in this way a specific point of the paper A'. The physical limitations of which we spoke are due to the fact that the acetate may extend beyond the paper, so that a point on the acetate does not cover a point on the paper, or the paper may extend beyond the acetate in which case there may be points on the paper not covered by a point on the acetate! We overcome these physical limitations by assuming that we can always extend the sheet of paper or the acetate. Abstracting this situation, we are led to the concept of a plane. In this new context, then, each point in the plane is covered by a specific point on the acetate in the initial position. In the final position, this point lies over a specific point in the plane. This assignment to each point in the plane of some specific point of the plane defines what we call a function from the set of points of the plane to the set of points of the plane. We will devote such a function by φ . In this case, the point of the plane assigned to the point V of the plane by φ is called the image of V and denoted $\varphi(V)$. A transformation is then a function which satisfies two extra conditions, namely:

(i) every point in the plane is the image of some point of the plane;

(ii) distinct points have distinct images.

The function defined by the acetate is in fact a transformation, for if W is any point, then in the final position of the acetate some point on the acetate lies over W. This point in the initial position lies over a point, say V, whose image is precisely W. Moreover, distinct points of the plane are covered by distinct points of the acetate in the initial position. In the final position, these cover distinct points of the plane.

The transformation defined in terms of the initial and final positions of the acetate is a motion of the plane. While this may seem like a very unmathematical definition, a more precise formulation would only provide a formal and probably less comprehensible description of what we have described!

Consider now three separate positions of the acetate--a first position, a second position, and a third position. Let φ denote the transformation with initial position of the acetate the first position, and final position, the second position. Similarly, ψ is the transformation with initial position the second position and final position the third position, and η is the transformation with the first position of the acetate sheet as initial position and the third position as final position.

These transformations φ, ψ, and η are related in a special way. If V is any point in the plane, let $\bar{V}$ be the point on the acetate which, in the first position covers V. This is the initial position for both φ and η. In the second position of the acetate $\bar{V}$ covers a specific point of the plane. This position is the final position for the transformation φ, so that $\bar{V}$ covers the image of V under φ, namely $\varphi(V)$. This second position is also the initial position for the transformation ψ. In the third position, $\bar{V}$ again lies over some specific point of the plane which is thereby the image under ψ of the point covered by $\bar{V}$ in the initial position for ψ, (the second position) namely $\varphi(V)$. Thus, in the third position, $\bar{V}$ covers $\psi(\varphi(V))$. However, since this third position is also the final position for the transformation η, $\bar{V}$ covers $\eta(V)$. Thus,

$$\eta(V) = \psi(\varphi(V))$$

This is true for every point of the plane. Summarizing, η assigns to the point V the point assigned by ψ to $\varphi(V)$, the image of V under φ. We express this relation by saying that η is the composition of φ followed by ψ and write

$$\eta = \psi\varphi$$

Note that the composition of the transformation φ followed by the transformation ψ is the transformation $\psi\varphi$.

We may represent this situation as follows:

$$\begin{tikzcd}
\text{Position 1} \arrow[r, bend left, "\varphi"] \arrow[rr, bend right, "\eta"'] & \text{Position 2} \arrow[r, bend left, "\psi"] & \text{Position 3}
\end{tikzcd}$$

Extending our considerations to a fourth position of the acetate sheet gives rise to three more transformations, as indicated below:

$$\begin{tikzcd}
\text{Position 1} \arrow[r, bend left, "\varphi"] \arrow[rr, bend right, "\eta"'] \arrow[rrr, bend right=50, "\omega"'] & \text{Position 2} \arrow[r, bend left, "\psi"] \arrow[rr, bend right, "\theta"'] & \text{Position 3} \arrow[r, bend left, "\chi"] & \text{Position 4}
\end{tikzcd}$$

As before, we can easily see that ω is the composition of η followed by χ since $\bar{V}$ (in the acetate) covers V at position 1, $\eta(V)$ at position 3 and $\omega(V)$ at position 4. But it also covers $\chi(\eta(V))$ whence

$$\omega = \chi\eta$$

Similarly,

$$\omega = \theta\varphi$$

So that

$$\begin{aligned} \omega &= \chi\eta &&= \chi(\psi\varphi) \\ &= \theta\varphi &&= (\chi\psi)\varphi \end{aligned}$$

which we usually write as

$$\chi(\psi\varphi) = (\chi\psi)\varphi$$

This, of course, expresses the associativity of composition of motions (or transformations).

Finally, suppose a motion φ has position 1 as initial position and position 2 as final position--the motion with position 2 as initial position and position 1 as the final position is then related to φ in a very special way. If V, any point in the plane, is covered by $\bar{V}$ in the initial position of φ (position 1), then in the final position of φ (position 2), $\bar{V}$ covers $\varphi(V)$. This new motion then assigns to $\varphi(V)$ the point V, and hence, undoes the effect of φ. This is called the inverse of φ and is denoted φ^{-1}. We then have, for any point V in the plane:

$$\varphi^{-1}(\varphi(V)) = V$$

The identity ι of the plane assigns to each point, itself so that

$$\iota(V) = V \quad \text{for every point } V.$$

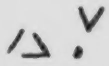
Thus, $$\varphi^{-1}\varphi = \iota$$

Summarizing, we regard the plane as an extendable sheet of paper which is "duplicated" by an extendable acetate sheet. There are two positions of this acetate on the paper which we designate as the initial position and the final position, respectively. Any point V in the plane is covered by one and only one point of the acetate. If, in the initial position, the point $\bar{V}$ on the acetate covers the point V on the paper, and in the final position of the acetate, the point $\bar{V}$ covers the point V' of the plane (the extended sheet of paper), the transformation φ, defined by these initial and final positions, maps the point V to the point V'!

Transformations defined in this way, in terms of an initial and a final position of the acetate, are called motions. The composition of two motions is again a motion, with initial position, the initial position of the first motion, and final position, the final position of the second motion. Composition of motions defines a binary operation in the set $\mathfrak{m}$ of motions. Since composition of functions, and hence motions, is associative, the identity is a motion and the inverse of a motion is a motion, $\mathfrak{m}$ is a group under the binary operation of composition of motions. In this way, we have defined a geometry in the plane!

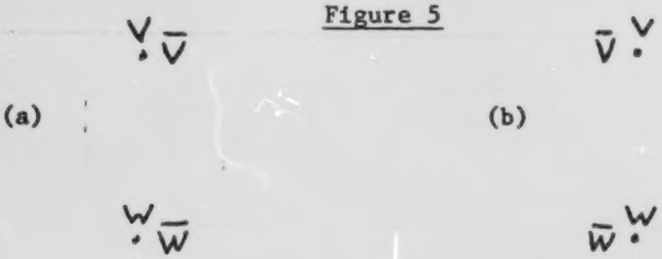
One problem which we have sidestepped is that of specifying the initial and final positions of the acetate. Specifically, if the acetate is moved in any way, how can we be sure of replacing it in the original position? Thus, if V is a point in the plane covered in the initial position by the point $\bar{V}$ on the acetate, and $\bar{V}$, in final position, covers the point V', then V' is the image of V under the motion φ determined by this initial and this final position. However, how do we find the image of a point W under this motion φ? We know that in the initial position the point $\bar{V}$ on the acetate covers the point V on the plane, but there are many ways to replace the acetate so that the point $\bar{V}$ covers the point V (see figure 4).

Figure 4



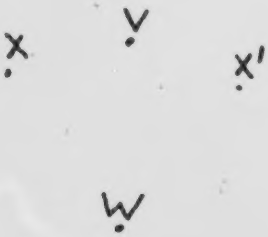
How do we determine the correct position? In particular, if the acetate is replaced so that $\bar{V}$ covers V, the acetate is free to rotate about this point $\bar{V}$ (imagine a pin through the point $\bar{V}$ and the point V). One way to eliminate this rotation is to stick another pin through the acetate and the paper at another point. That is, choose a point W in the plane different from the point V and let the point $\bar{W}$ on the acetate cover the point W on the plane. This certainly restricts our freedom considerably, but still leaves us with two choices for placing the acetate on the paper so that the point $\bar{V}$ covers the point V and the point $\bar{W}$ covers the point W. For if we have already done this in one way (see figure 5a), then we can turn the acetate over and replace it so that the point $\bar{V}$ again covers the point V and the point $\bar{W}$ again covers the point W (see figure 5b)!

Figure 5



In order to distinguish these two positions, choose a point $\bar{X}$ on the acetate not on the line $\bar{V}\,\bar{W}$. Now the line $\bar{V}\,\bar{W}$ lies over the line in the plane V W which separates the plane into two half-planes. In one position of the acetate the point $\bar{X}$ lies over a point X in one of these half-planes, and in the other flipped position the point $\bar{X}$ lies over a point X' in the other half-plane (see figure 6).

Figure 6



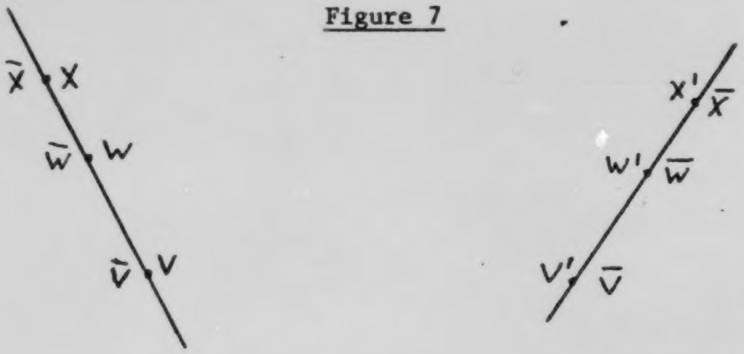
Choosing one of these points, say X, there is only one way we can place the acetate so that the point $\bar{V}$ covers the point V, the point $\bar{W}$ covers the point W, and the point $\bar{X}$ covers the point X. Thus, we can completely determine or fix a position of the acetate by choosing three non-collinear points in the plane V, W, X and their surrogate points in the acetate $\bar{V}$, $\bar{W}$ and $\bar{X}$.

Now a motion is defined in terms of an initial and a final position of the acetate. Any position of the acetate may be fixed by choosing three non-collinear points in the plane and their surrogate points in the acetate.

It seems reasonable, then, to choose three non-collinear points in the plane, V, W, X. In the initial position of the acetate, the points $\bar{V}$, $\bar{W}$ and $\bar{X}$ in the acetate cover the points V, W and X respectively; in the final position, the point $\bar{V}$ lies over the point V', $\bar{W}$ over W', and $\bar{X}$ over X'. Thus, the non-collinear points V, W and X and their images under the motion, V', W' and X' respectively, completely determine or fix the initial and final positions of the acetate and hence, completely determine the motion. This assertion is in fact an important theorem with profound consequences. However, before going into any of them, we first consider the question of invariant properties under the group of motions.

Let the points V and W be any points in the plane. In the initial position of the acetate the points V and W are covered by their surrogate points $\bar{V}$ and $\bar{W}$ in the acetate. If, now, X is any point on the line VW in the plane, then its surrogate point $\bar{X}$ will lie on the line $\overline{VW}$ on the acetate. In the final position of the acetate the line $\overline{VW}$ will cover the line V'W', where the point V' is the image of the point V and the point W' is the image of the point W under the given motion. Hence, the point X', the image of the point X, is covered by the surrogate point $\bar{X}$ in this final position (see figure 7) so that X' must lie on the line V'W'.

Figure 7



A set of points in the plane is collinear if for some line, all the points in the set lie on this line.[*] This property of a set, called collinearity, is invariant under a motion.

If the point X is chosen between the points V and W (so that in moving along the line from the point V to the point W we must pass the point X), then its surrogate point $\bar{X}$ on the acetate will lie between the points $\bar{V}$ and $\bar{W}$. Hence, the point X', the image of the point X under the motion, will be covered by the surrogate point $\bar{X}$ and therefore must lie between the points V' and W'. Thus, betweeness is invariant under motions. Hence, the line segment $\overline{VW}$ is mapped onto the line segment $\overline{V'W'}$ and line segments are invariant. Similarly, lines are mapped onto lines and rays onto rays under a motion.

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[*] Note that in order to define a collinear set we must have some "structure" in the space (the plane), namely lines.

If V and W are two points in the plane, then the distance between their surrogates $\bar{V}$ and $\bar{W}$ is the same as that between V and W, and also, therefore, in the final position of the acetate, the same as the distance between the image points V' and W' of the points V and W. Hence, distance is invariant under a motion.

If two lines in the plane are parallel then their surrogates on the acetate are also parallel. That is, parallel lines in the plane are covered by parallel lines in the acetate so that the image of a pair of parallel lines under a motion will again be a pair of parallel lines so that parallelism is an invariant under the group of motions.

Since the surrogate or tracing on the acetate of an angle in the plane is again an angle of the same measure, angles are mapped into angles under motions, and the measure of an angle is invariant under any motion.

Similarly, a circle is mapped into a circle, and ellipse into an ellipse, a hyperbola into a hyperbola, and a parabola into a parabola.

A set is convex if the line segment between any two points in the set is a subset of the set. Since line segments are invariant under a motion, so is convexity.

This raises the question--is there anything that isn't invariant under any motion? The position of a figure is not invariant under every motion nor are the attitude or orientation of the figure. Thus, the square in figure 8 can be mapped under a motion into the "diamond" in figure 8, and the "A" in figure 9 can be mapped into an upside-down "A." These are changes in attitude.

Figure 8

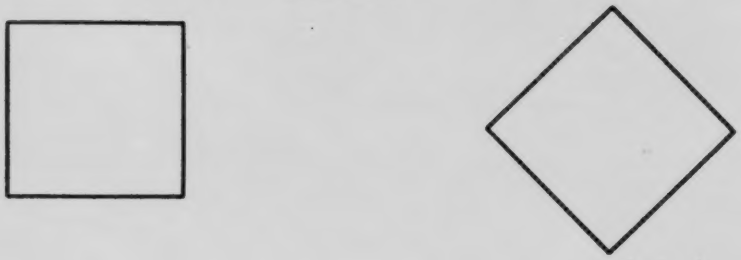
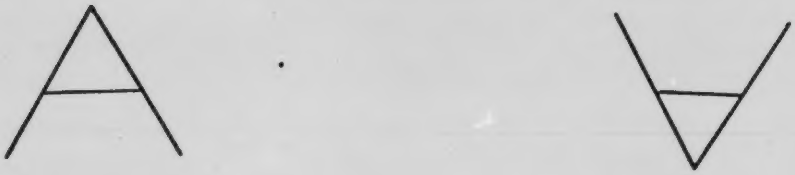
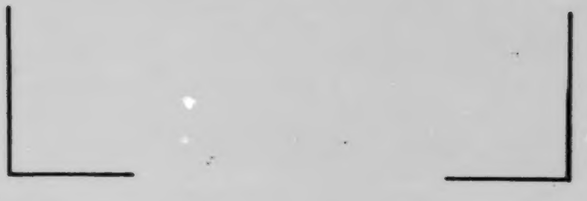


Figure 9



The notion of orientation is a little more complicated. The "L" in figure 10 can be mapped by a motion into the backwards or reversed "L" in figure 10. This represents a change in orientation. Recall that a line separates the plane into two half-planes. Now choose a direction along the line. This can be done by choosing two points V and W on the line so that the direction on the line is from the point V to the point W. If one is "standing" on the line facing in the direction of the line, then one of the half-planes is on the left and the other is on the right.

Figure 10



In figure 6a the point X is in the right half-plane. This choice of a direction along the line, in terms of two points (V and W) on the line, and one of the half-planes, (the right or the left half-plane) determines an orientation.

Figure 6a represents the initial position of a motion. Figure 6b represents the final position of the motion where the acetate is turned over and replaced so that the point $\bar{V}$ again covers the point V and the point $\bar{W}$ again covers the point W. The point $\bar{X}$ now covers the point X', the image of the point X. However, X' is in the other half-plane from X! Thus, going from the point V to the point W, X is in the right half-plane, and X' is in the left half-plane. This motion reverses the orientation.

With regard to the "L" of figure 10, if we move from the corner along the longer arm of the "L" (the vertical arm) this defines a direction along the line containing this arm and the short arm of the "L" is in the right half-plane. However, with regard to the reversed "L", moving again from the corner along the longer arm, the short arm is now in the left half-plane.

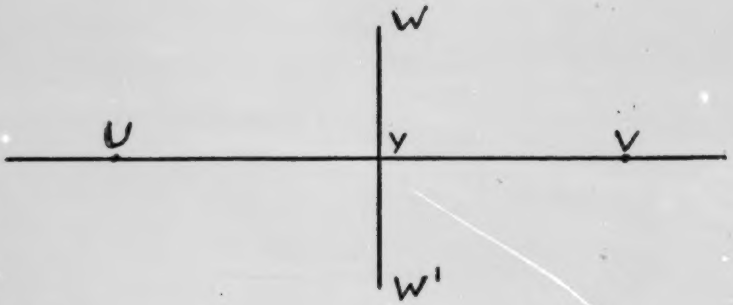
Alternatively, if we move out from the corner along the shorter arm, then the longer arm is on the left for the "L" and on the right for the reversed "L", that is, on opposite sides.

We express this by saying that they are oppositely oriented, so that a motion can reverse orientations and orientation is not invariant under all motions.

Let U and V be distinct points in the plane. Covering the plane with the acetate, let $\bar{U}$ and $\bar{V}$ be the points covering the points U and V in the plane, respectively. As we have already observed, there are two positions of the acetate with the point $\bar{U}$ covering the point U and the point $\bar{V}$ covering the point V. Choosing W a point in the plane not on the line U V, and its surrogate point $\bar{W}$ in the acetate, fixes one of these positions. Flipping the acetate and replacing it so that the point $\bar{U}$ again covers the point U and the point $\bar{V}$ again covers the point V, then $\bar{W}$ covers a point W' in the opposite half-plane of the line U V from the point W. If the position determined by the points U, V, and W is taken as the initial position and the position determined by the points U, V, and W' as the final position, a motion φ is determined. The image of the point U under φ is again the point U and the image of the point V under φ is again the point V. Indeed, since the line U V is mapped into itself, any point X on this line is mapped into the same point X. We express this fact by saying the line U V is point-wise fixed by this motion φ.

Since the points W and W' are on opposite sides of the line U V, the line segment $\overline{WW'}$ intersects the line at some point Y between the points W and W' (see figure 11).

Figure 11



Now the motion φ maps the point W onto the point W' and fixes the point Y, hence it maps the line segment $\overline{W\ Y}$ onto the line segment $\overline{W'\ Y}$-- thus Y is the midpoint of the segment $\overline{WW'}$. Moreover, since φ fixes U, it maps the angle $<$ U Y W onto the angle $<$ U Y W'. These angles therefore have the same measure, and since they are supplementary angles (the angle $<$ W Y W' is a straight angle!), each is a right angle. From this it follows that the line U V is the right bisector (or perpendicular bisector) of the line segment $\overline{W\ W'}$. Since W was any point not on the line U V, φ maps any point not on the line U V into the point W' such that U V is the right bisector of the line segment $\overline{W\ W'}$. Since it is also the right bisector of the line segment $\overline{W'\ W}$ (the same line segment as $\overline{W\ W'}$) φ maps the point W' to the point W. Thus

$$\varphi\varphi = \iota$$

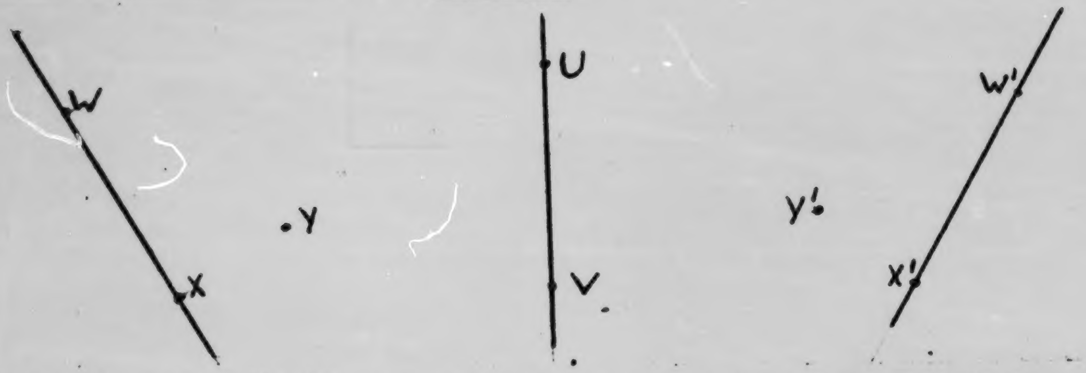
and

$$\varphi = \varphi^{-1}$$

The motion φ defined in this way from the line U V is called the reflection in the line U V.

Consider figure 12. U V is a line. Let W and X be distinct points and denote by W' and X' the images of the points W and X , respectively,

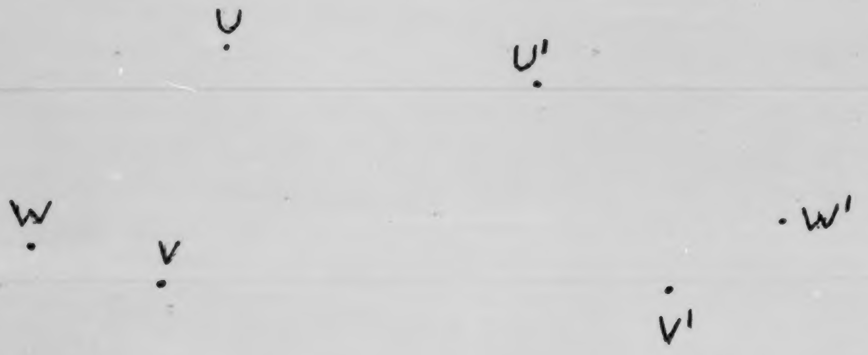
Figure 12



under φ , the reflection in the line U V. Then W' and X' are distinct points and the reflection φ maps the line W X onto the line W'X'. Let Y be a point not on the line W X as in figure 12, and let the point Y' be the image of Y under the reflection φ . Orienting the line W X from W to X, the point Y is in the left hand half-plane of the line W X. However, going from the point W' to the point X', the point Y' is in the right half-plane for the line W'X'. Thus, a reflection reverses orientation.

Now consider an arbitrary motion φ . If U, V, and W are three non-collinear points and the image under φ of the point U is the point U', of the point V is the point V', and of the point W is the point W', then φ is completely determined. That is, if in the initial position of the acetate the point $\bar{U}$ covers the point U, the point $\bar{V}$ covers the point V, and the point $\bar{W}$ covers the point W, then in the final position of the acetate, the point $\bar{U}$ covers the point U', the point $\bar{V}$, the point V', and the point $\bar{W}$, the point W'. In this way, the initial and final positions of the acetate are fixed and the motion determined (see figure 13).

Figure 13



If U and U' are distinct let α denote the reflection in the right bisector of the line segment $\overline{UU'}$. If $U = U'$, then α will denote the identity transformation. In either case, α maps the point U onto the point U', and we denote the image under α of the point V by V'' and of the point W by the point W''. (Of course, if α is the identity transformation then $V'' = V$ and $W'' = W$!).

If the point V'' is different from the point V', then denote by β the reflection in the right bisector of the segment $\overline{V'V''}$. Now since the line segment $\overline{U'V''}$ is congruent to the line segment $\overline{UV}$ and this latter line segment is congruent to the line segment $\overline{U'V'}$, the segments $\overline{U'V''}$ and $\overline{U'V'}$ are congruent so that the triangle $\triangle V''U'V'$ is isosceles and the right bisector of the base $\overline{V''V'}$ passes through the vertex U'. Hence, β maps the point U' onto itself. If $V'' = V'$ then β denotes the identity transformation. In either case, β maps the point U' onto itself and the point V'' onto the point V'. Let W''' denote the image of the point W'' under the motion β. (Again, if $\beta = \iota$, then $W''' = W''$.)

Finally, if the point W''' is distinct from the point W', then as before, $\triangle W'''U'W'$ and $\triangle W'''V'W'$ are both isosceles with base the segment $\overline{W'''W'}$ and the right bisector of this segment passes through the vertex U' of the triangle $\triangle W'''U'W'$ and through the vertex V' of the triangle $\triangle W'''V'W'$. In this case (where the points W''' and W' are distinct), let γ denote the reflection in the line $U'V'$. Otherwise (if $W''' = W'$) then γ will denote the identity transformation. In either case, γ fixes the points U' and V' and maps the point W''' onto the point W'. Thus, since

$$\alpha(U) = U', \quad \beta(U') = U', \quad \gamma(U') = U'$$

it follows that

$$\gamma(\beta(\alpha(U))) = U'$$

Similarly,

$$\alpha(V) = V', \quad \beta(V') = V'' \text{ and } \gamma(V'') = V'$$

so that

$$\gamma(\beta(\alpha(V))) = V'$$

and

$$\alpha(W) = W'', \quad \beta(W'') = W''', \quad \gamma(W''') = W'$$

so that

$$\gamma(\beta(\alpha(W))) = W'.$$

Thus, since the motion $\gamma\beta\alpha$ maps the point U onto the point U', the point V onto the point V', and the point W onto the point W' ~~so that~~ the initial and final positions of the acetate for the motion $\gamma\beta\alpha$ are exactly the same as those for the motion φ. Hence,

$$\gamma\beta\alpha = \varphi$$

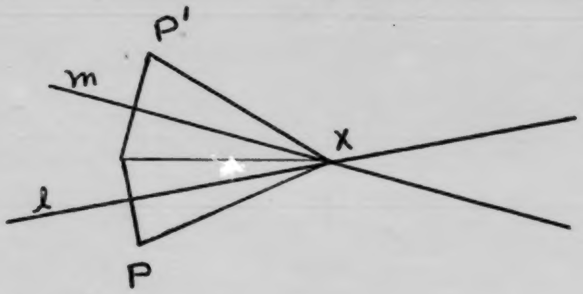
and we have shown that any motion can be represented as the composition of three motions, each of which is either the identity motion or a reflection. Since the identity motion may be deleted from the above representation without changing anything, we have shown that any motion may be represented as the composition of at most three reflections. This means that the group of motions is generated by the reflections!

Recall that a reflection reverses orientation so that the composition of two reflections will leave the orientation unchanged or invariant. Thus if the motion φ reverses the orientation and is not a reflection then it must be the composition of exactly three reflections. (This is the case illustrated in figure 13!) If orientation is invariant under φ, then φ must be the identity or the composition of exactly two reflections. Such a motion, which leaves the orientation invariant, is called a direct motion.

Since a direct motion can be represented as the composition of two reflections, the composition of two direct motions can be represented as the composition of four reflections. Each of these reflections reverses the orientation, giving rise to four reversals. The net result is, therefore, to leave the orientation unchanged. Hence the composition of two direct motions is again a direct motion. Moreover, the inverse of a direct motion is also a direct motion. It follows from this that the direct motions form a subgroup $\mathcal{M}_d$ of the group of motions $\mathcal{M}$ and hence defines a subgeometry in which orientation is an invariant.

A direct motion φ can be represented as the composition of two reflections--that is, reflections in two lines. We distinguish two cases. In the first case, the lines intersect at the point X (see figure 14). Then X is left fixed by each of the reflections and hence by their composition. If P is any other point and the image of the point P under this motion φ is P', then the motion φ maps the segment $\overline{PX}$ onto the segment $\overline{P'X'}$. The first line ℓ can be rotated about the point X through some angle onto the second line m. Rotating the segment $\overline{PX}$ about X in the same direction but through twice the angle will bring the segment $\overline{PX}$ onto $\overline{P'X}$ (see figure 14). Hence we describe this motion as a rotation about X.

Figure 14



In the second case, the lines ℓ and m are parallel (see figure 15). Then, if P is any point in the plane and P' is its image under this motion, the segment $\overline{PP'}$ is perpendicular to each of the lines ℓ and m and its length is twice the distance between the lines. If Q is another point in the plane and Q is not on the line P P', then P Q Q' P' is a parallelogram so that P Q is parallel to P' Q'. On the other hand, if Q is on the line P P', then so is its image Q'. In either case, this motion maps a line into a parallel line. That is, the attitude is invariant under this motion. Such a motion is called a translation.

If ℓ, m and n are three lines with the line m perpendicular to both the lines ℓ and n, (so that the lines ℓ and n are parallel), then composition of the reflection in the line ℓ followed by the reflection in the line m is a rotation about X, their point of intersection. Similarly, the composition of the reflection in the line m followed by the reflection in the line n is also a rotation, but about the point Y (see figure 16).

Figure 15

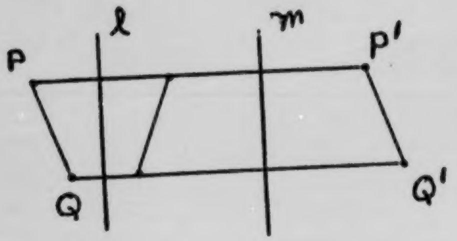
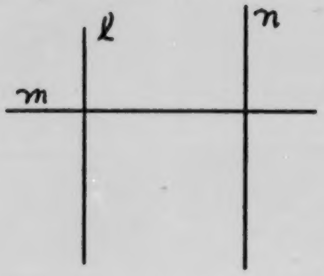


Figure 16



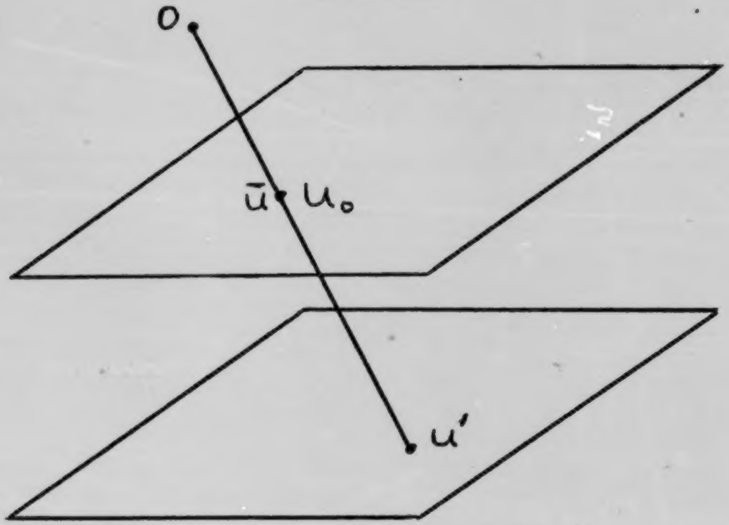
The composition of these two rotations is the composition of the reflection in the line ℓ followed by the reflection in the line n. Since these lines are parallel, this composition is a translation and not a rotation! On the other hand, the composition of two translations is again a translation, so that the translations form a subgroup $\mathcal{T}$ of the group of direct motions $\mathcal{M}_d$. The subgeometry determined by $\mathcal{T}$ has as invariants, attitude as well as orientation.

The Group of Similarities

In discussing motions, the initial and final positions of the acetate served primarily as a means of describing the assignment to a point U in the plane of a point U' in the plane, the image of U under the motion defined by this initial and final position of the acetate. Indeed, it is precisely this assignment to every point of the plane of a point of the plane which is the transformation. The acetate, as a surrogate for the plane, can be used to introduce and describe more general transformations.

The first class of these will be defined as follows: the initial position of the acetate will be, as before, on the paper; however, the final position will be in space parallel to the plane of the paper. In this way, the surrogate of a point U in the plane, namely the point $\bar{U}$ on the acetate, covers the point U in the initial position. However, in the final position $\bar{U}$ covers some point U_o in space. In order to get a point in the plane, choose a fixed point O in space such that O is not in the plane of the paper and not in the plane of the acetate when in the final position. The line O U_o will meet the acetate in the final position at the point $\bar{U}$ and the plane of the paper at the point U'. The transformation defined by this initial and final position of the acetate assigns to the point U in the plane the point U' in the plane. Transformations defined in this way are called similarities (see figure 17).

Figure 17



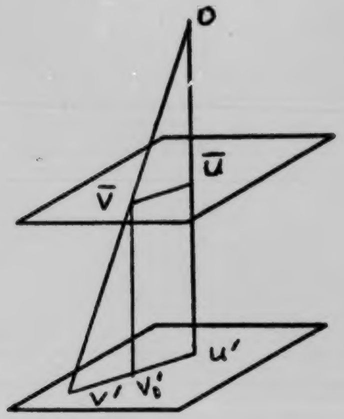
The point O in the definition of a similarity is called the center of projection. Any line through O not parallel to the plane of the paper will meet the acetate, in its final position, at a point say $\bar{X}$ and meet the plane at a point X'. The point X' is the projection of the point $\bar{X}$ from the center O.

In order to avoid confusion with other planes which will be introduced in the discussion that follows, we denote the plane (of the paper) which we are studying by π .

Consider now a similarity φ defined as above by an initial position of the acetate in the plane π , a final position in space parallel to the plane, and a center of projection O. In order to facilitate the determination of the invariant properties under similarities, we first represent any similarity, say φ , in terms of more special transformations.

The line through O perpendicular to the plane (π) will project the point $\bar{U}$ on the acetate into the point U' in the plane. The point $\bar{U}$ on the acetate will cover a point U in the plane in its initial position, and a point Uo in space in its final position. The similarity φ maps the point U onto the point U' (see figure 18).

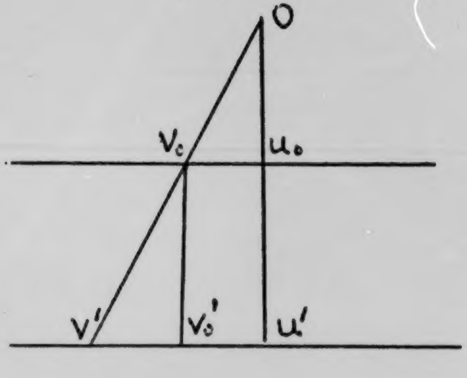
Figure 18



Let $\bar{V}$ be another point in the acetate covering the point V in the plane, in the initial position of the acetate, and the point Vo in space in the final position of the acetate. The line O Vo meets the plane in a point V', the projection of $\bar{V}$ from O and the image of V under φ .

Since the line O Uo is perpendicular to the plane, the plane O Uo Vo is also perpendicular to the plane of the paper, π (see figure 19). The sheet of paper of the figure represents the plane O Uo Vo and we "see" the acetate and plane of the paper, π , from "the edge." That is, their intersections with the acetate on the plane of the acetate in its final position and with the plane of the paper π .

Figure 19



The line through the point Vo parallel to the line O Uo will be perpendicular to the plane (of the paper) π , and meet this plane at a point Vo'. Moreover, the point Vo' will lie on the line U' V' in the plane. Now Uo U' Vo' Vo is a rectangle so that the line segment $\overline{U'\ Vo'}$ is congruent to the line segment $\overline{Uo\ Vo}$ and hence to the line segments $\bar{U}\ \bar{V}$ and $\overline{U\ V}$. If W is another point in the plane with surrogate $\bar{W}$ on the acetate, covering, in the final position, the point Wo in space, let the line O Wo meet the plane in the point W'. Then W' is the image of W under this similarity φ . Now, since the line Uo Vo is parallel to the line U' V', the triangles Δ O Uo Vo and Δ O U' V' are similar so that

$$\frac{|\ O\ Uo\ |}{|\ O\ U'\ |} = \frac{|\ O\ Vo\ |}{|\ O\ V'\ |} = \frac{|\ Uo\ Vo\ |}{|\ U'\ V'\ |} = \frac{|\ U'\ Vo'\ |}{|\ U'\ V'\ |}$$

Moreover, since the line Uo Wo is parallel to the line U' W' and the line Vo Wo is parallel to the line V' W', the triangles Δ O Uo Wo and Δ O U' W' are similar as are the triangles Δ O Vo Wo and O V' W'. Hence

$$\frac{|\ O\ Uo\ |}{|\ O\ U'\ |} = \frac{|\ O\ Wo\ |}{|\ O\ W'\ |} = \frac{|\ Uo\ Wo\ |}{|\ U'\ W'\ |} = \frac{|\ U'\ Wo'\ |}{|\ U'\ W'\ |}$$

and

$$\frac{|\ O\ Vo\ |}{|\ O\ V'\ |} = \frac{|\ O\ Wo\ |}{|\ O\ W'\ |} = \frac{|\ Vo\ Wo\ |}{|\ V'\ W'\ |} = \frac{|\ Vo'\ Wo'\ |}{|\ V'\ W'\ |}$$

It follows, therefore, that for any two points V and W in the plane, since

$$|\ V\ W\ | = |\ \bar{V}\ \bar{W}\ | = |\ Vo\ Wo\ |$$

that

$$\frac{|\ V\ W\ |}{|\ V'\ W'\ |} = \frac{|\ Vo\ Wo\ |}{|\ V'\ W'\ |} = \frac{|\ O\ Uo\ |}{|\ O\ U'\ |}$$

so that, while a similarity will not, in general, leave distances invariant, for each similarity there is a fixed number k ($\neq 0$) by which all distances are multiplied. This is the magnification factor.

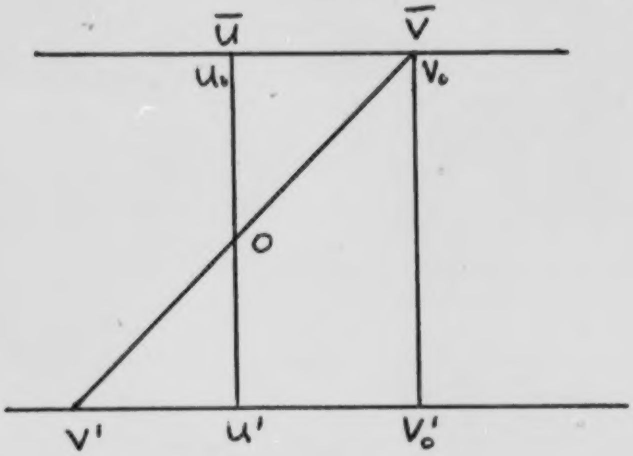
If now the point W were chosen not on the line U V, the three points U, V, and W would then serve to fix the initial position of the acetate. Similarly, the points Uo, Vo, and Wo serve to fix the final position of the acetate in space. Now, if the acetate were "dropped" into the plane π along the lines Uo U', Vo Vo', Wo Wo', the point $\bar{U}$ on the acetate would now cover the point U', the point on the acetate $\bar{V}$ would now cover the point Vo' in the plane and the point $\bar{W}$ could cover the point Wo' in the plane. In this way, the points U', Vo', Wo' serve to fix an intermediate position of the acetate in the plane. Taking the initial position of the acetate for the similarity φ as the initial position, and this intermediate position of the acetate as the final position, defines a motion which we denote by μ_0. Similarly, this intermediate position as initial position of the acetate, and the final position of the acetate for φ as final position determines a similarity which we denote by δ_0. Then,

$$\delta_0 \mu_0 = \varphi ,$$

that is, the similarity φ may be represented as the composition of a motion μ_0 followed by a similarity δ_0. This similarity is quite special. It leaves fixed the point U' and maps any point Vo' distinct from U' onto the point V' on the line U' Vo' and such that the distance from V' to U' is some fixed constant k times the distance from U' to Vo'.

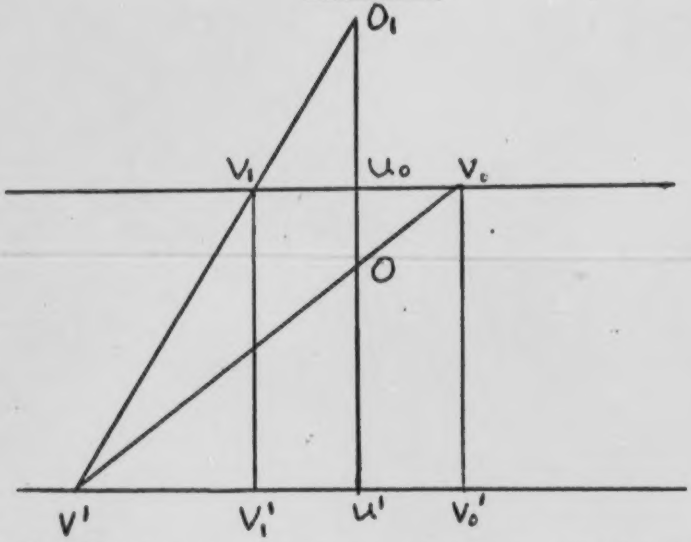
Considering this similarity δ_0, we must examine some separate cases. If the center of projection O is not between Uo and U', say Uo is between O and U' or U' is between O and Uo, then a point Vo' is mapped onto the ray $\overrightarrow{O\ Vo'}$. However, if O is between U' and Uo, then the point Vo' is mapped into the opposite ray so that O is between Vo' and V' (see figure 20).

Figure 20



In this latter case (where O is between the points Uo and U') a half-turn about U' (a rotation of 180°) will map the point Vo' into a point V_1' on the ray $\overrightarrow{U'\ V'}$ (see figure 21). If the point V_1' coincides with the point V', then the magnification factor is 1 and this half-turn defines the same transformation of the plane as is defined by δ_0 so that φ is in fact a motion! If $k \neq 1$ then the point V_1' will be different from the point V' and the line through the point V_1' parallel to O U' will meet the acetate (in the final position for φ) at a point $\overline{V}_1$ covering the point V_1 in space. Now V_1 is on the line Uo Vo but with Uo the midpoint of the segment $\overline{Vo\ V_1}$.

Figure 21



The line V_1 V' will lie in the plane O Uo Vo and will meet the line O Uo in a point O_1. Since the line Uo V_1 is parallel to the line U' V', the two triangles Δ O_1 Uo V_1 and Δ O_1 U' V' are similar. Thus,

$$\frac{|O_1 \quad U_0|}{|O_1 \quad U'|} = \frac{|U_0 \quad V_1|}{|U' \quad V'|} = \frac{|U_0 \quad V_0|}{|U' \quad V'|} = \frac{|\ OU_0\ |}{|\ OU'\ |} = k$$

Similarly, let W_1' be the image of Wo' under the half-turn about U'. The point $\bar{W}_1$ on the acetate covering the point W_1 in space is the unique point such that the line W_1 W_1' is parallel to Uo U'. Once again, the projection of W_1 from O_1 is W' . If now we take as the initial position of the acetate the initial position for the similarity φ and for the final position that determined by the points in space, U_0 , V_1, W_1 (that is a half-turn of the acetate about the line Uo U' from the previous final position) and as the center of projection O_1 instead of O, then we have another way of defining the similarity φ , for any point V in the plane is mapped in this way into the same V'. That is, the point V' in the plane assigned to the point V of the plane is the same for either definition and hence both determine the same similarity φ ! With this new description we can represent φ as the composition of

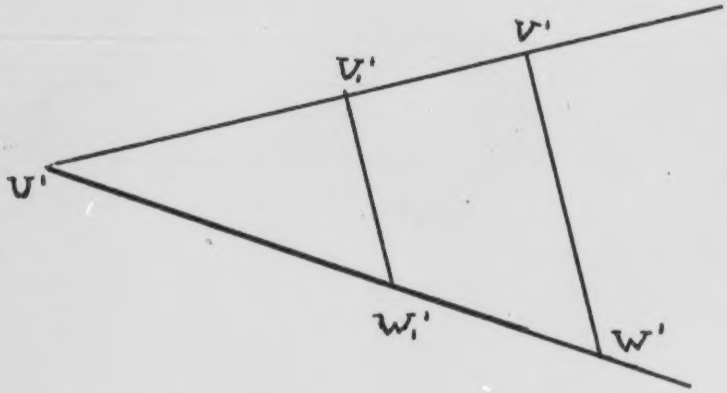
$$\delta_1 \mu_1 = \varphi$$

where μ_1 is the motion and δ_1 a similarity determined by the intermediate position of the acetate where the point $\bar{U}$ covers the point U', the point $\bar{V}$ covers the point V_1', and the point $\bar{W}$ covers the point W_1'. The motion μ_1 is simply the motion μ_0 followed by a half-turn about U' and δ_1 is the similarity which fixes U' and maps any point V' different from U' onto a point V' on the ray $\overrightarrow{U' V_1'}$ such that

$$|U'V'| = k|U'V_1'|$$

where k is the magnification factor (see figure 22). A similarity with the properties of δ_1 above is called a homothety of center U' and magnification k.

Figure 22



Thus, what we have shown is that any similarity can be represented as the composition of a motion followed by a homothety.

One can show without too much difficulty that the similarities, including motions, form a group generated by the motions and the homotheties. However, our main interest is in the invariant properties under this group, and since any similarity not a motion can be represented as the composition of a motion and a homothety, we need only consider which properties, invariant under motions, are invariant under homotheties.

Consider, therefore, a homothety δ with center C and magnification k. Let U be any point in the plane. Then δ maps U into the point U' on the ray $\overrightarrow{CU}$ such that $|CU'| = k|CU|$ (see figure 23). Let V be another point.[*] If V is not on the line C U then the line through the point U' parallel to the line U V will meet the ray $\overrightarrow{CV}$ at a point V' and the triangles Δ C U V, Δ C U' V' are similar. Since

$$\frac{|CU'|}{|CU|} = k$$

---

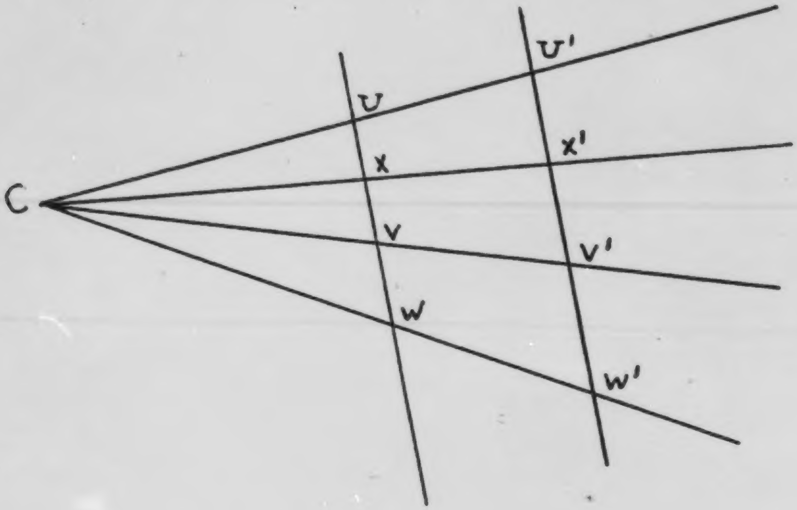
[*] If V is on the line C U then so is V'.

it follows that

$$\frac{|CV'|}{|CV|} = k$$

and the point V' must be the image of the point V under the homothety δ .

Figure 23



The same argument will show that if W is any point on the line U V, then the image of the point W, namely, the point W', will lie on the line U' V'. Thus, collinearity is invariant under a homothety. Moreover, if the point X is between the points U and V then the point X' is between U' and V' (see figure 23) so that betweenness is invariant under a homothety. Thus, the homothety maps a line segment $\overline{UV}$ onto the line segment $\overline{U'V'}$ and a ray $\overrightarrow{UV}$ is mapped onto the ray $\overrightarrow{U'V'}$. From the above discussion we see that a line is mapped onto a parallel line so that attitude (or direction) is invariant under a homothety. Distance is not invariant under a homothety but is multiplied by the magnification factor.

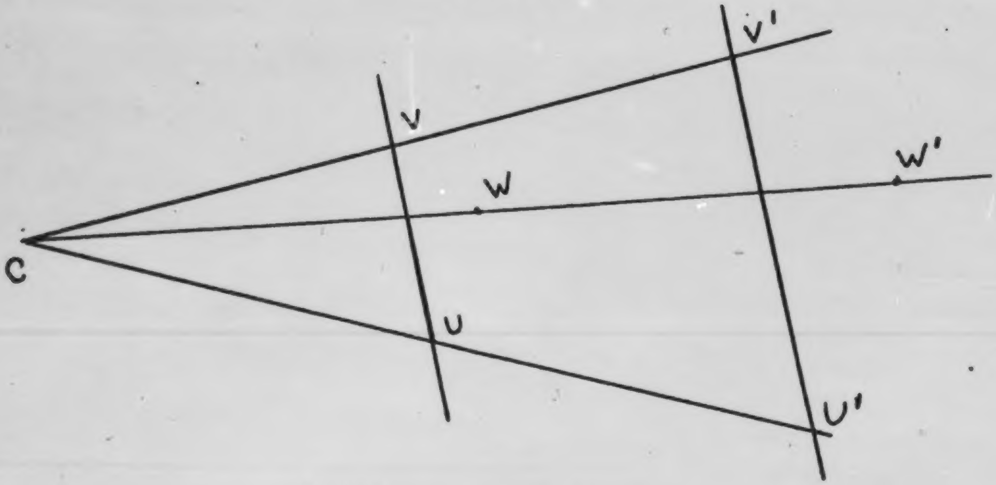
Since a homothety maps a line onto a parallel line, parallel lines are mapped into parallel lines and parallelism is invariant under a homothety.

Consider the angle $\angle UVW$. Since the ray $\overrightarrow{VU}$ is mapped into the ray $\overrightarrow{V'U'}$ and the ray $\overrightarrow{VW}$ is mapped into the ray $\overrightarrow{V'W'}$ the angle $\angle UVW$ is mapped into the angle $\angle U'V'W'$. In view of the fact that the line V' U' is parallel to the line V U and the line V' W' is parallel to the line V W, the measure of the angle is invariant.

Similarly, circles are mapped into circles, ellipses into ellipses, hyperbolae into hyperbolae and parabolae into parabolae.

Convexity is also invariant since intervals are invariant. Finally, orientation is also invariant (see figure 24).

Figure 24



Let $\mathcal{S}$ denote the group of similarities. Then $\mathcal{S}$ leaves invariant collinearity, betweenness, line segments, rays, lines, parallelism, angles, and the measure of angles, shapes (circles, ellipses, parabolae and hyperbolae) and convexity. If $\mathcal{S}_d$ is the group of direct similarities generated by the direct motions and the homotheties then $\mathcal{S}_d$ leaves orientation invariant. Finally, if $\mathcal{S}_T$ is the subgroup of direct similarities generated by the translations and the homotheties, then $\mathcal{S}_T$ leaves attitude or direction invariant, since a line is mapped onto a parallel line.

Distance is not invariant, but for each similarity there is a magnification factor k for distances. Let U, V, W, X be points in the plane, and let their respective images under a similarity be the points U', V', W', and X'. Then

$$|U'V'| = k|UV|$$

$$|W'X'| = k|WX|$$

where k is the magnification of the similarity. Then

$$\frac{|U'V'|}{|W'X'|} = \frac{k|UV|}{k|WX|} = \frac{|UV|}{|WX|}$$

Thus, while distances are not invariant, the ratio of two segments is invariant!

A similarity maps a square into a square of edge k times the edge of the original square. Hence the similarity multiplies the area of the square by a factor of k^2. Similarly, for any figure, the area of the image is k^2 times the area of the figure. However, if $\underline{F}$ and $\underline{G}$ are two figures with areas $|\underline{F}|$ and $|\underline{G}|$ respectively and if $\underline{F}'$ and $\underline{G}'$ are their images under a given similarity,

$$|\underline{F}'| = k^2 |\underline{F}|$$

$$|\underline{G}'| = k^2 |\underline{G}|$$

and

$$\frac{|\underline{F}'|}{|\underline{G}'|} = \frac{k^2 |\underline{F}|}{k^2 |\underline{G}|} = \frac{|\underline{F}|}{|\underline{G}|}$$

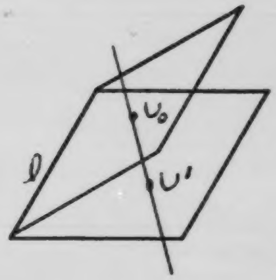
so that the ratio of the area of two figures is invariant under similarities.

The Affine Group

In this section we consider transformations defined by an initial position of the acetate in the plane and a final position of the acetate inclined at an angle to the plane rather than parallel to it. To complete the description, we must indicate how to get a point in the plane corresponding to a point on the acetate! We do this by first fixing a direction in space not parallel to either the plane or the acetate in its final position. (This can be done by choosing a line which intersects both the plane and the acetate in its final position.) If U is a point in the plane and $\bar{U}$ its surrogate on the acetate, then $\bar{U}$ covers U in the initial position of the acetate and covers a point Uo in the final position. The line through Uo in the fixed direction (parallel to the chosen line) will meet the plane in the point U'. We say that U' is the projection of Uo along the given direction. The point U' is the image of the point U under the transformation defined in this way (see figure 25). A transformation defined in this way is called an affine transformation.

The acetate, in this final position, intersects the plane in a line ℓ (see figure 25).

Figure 25



If the point $\bar{U}$ is not on the line ℓ, then the point U' will not be on the line ℓ also. Starting with the acetate in the final position, rotate the acetate about the line ℓ until it is in the plane of the paper. Choose the direction of rotation so that $\bar{U}$ and U' are in the same half-plane of ℓ. (We may imagine a book open on the table, and a leaf of the book not flat but inclined at an angle.) The leaf represents the acetate, the table (or the open book), the plane, and the spine of the book the line ℓ. We can turn the leaf to the flat position by turning back or turning forward!) This defines an intermediate position of the acetate. We define a motion where the initial position of the acetate is the initial position for the affine transformation or affinity, and the final position is the intermediate position of the acetate described above. We define another affinity by taking as the initial position the intermediate position of the acetate and as final position, the final position of the acetate for the given affinity. In this way, the affinity is represented as the composition of a motion and one of these special affinities.

As with similarities, one can show that the composition of two affinities is again an affinity and the inverse of an affinity is an affinity so that the affinities form a group α. However, since we are concerned with the invariants under this group, we direct our attention to the special affinity described above.

In rotating the acetate about the line ℓ from its initial to its final position, points on ℓ remain fixed. Specifically, if $\bar{L}$ is a point on the acetate covering the point L on the line ℓ in the initial position, $\bar{L}$ also covers L in the final position. A line in the fixed direction through $\bar{L}$ intersects the acetate and the plane in the same point. Hence, the affinity fixes every point on ℓ. Such an affinity is called an axial affinity with axis ℓ.

Let U be a point in the plane with surrogate on the acetate, point $\bar{U}$, and let V be another point with surrogate $\bar{V}$ on the acetate. Then, in the initial position, the point $\bar{U}$ covers the point U, the point $\bar{V}$ covers the point V, and the line $\bar{U}\,\bar{V}$ covers the line U V. In the final position, the point $\bar{U}$ covers the point Uo, the point $\bar{V}$ covers the point Vo, and the line $\bar{U}\,\bar{V}$ covers the line Uo Vo. The line through Uo in the fixed direction and the line Uo Vo determine a plane which meets the plane in the line U' V' where the point U' is the image of U under the affinity and V' is the image of V. If W is any point on the line U V, then $\bar{W}$ is on the line $\bar{U}\,\bar{V}$ and covers W in the initial position and a point Wo on the line Uo Vo in the final position. The line in the fixed direction through Wo lies in the plane Uo Vo V', and hence, projects Wo into a point W' on the line U' V'. Moreover, if W is between U and V, then W' is between U' and V' (see Figure 26).

Thus, collinearity and betweenness are invariant under an axial affinity. Hence, lines, line segments, and rays are invariant so that an axial affinity maps the line segment $\overline{U\,V}$ onto the line segment $\overline{U'\,V'}$, the ray $\overrightarrow{UV}$ onto the ray $\overrightarrow{U'\,V'}$, and the line U V onto the line U'V'. It follows, therefore, that angles are mapped onto angles and convex sets onto convex sets; that is, angles and convexity are invariant.

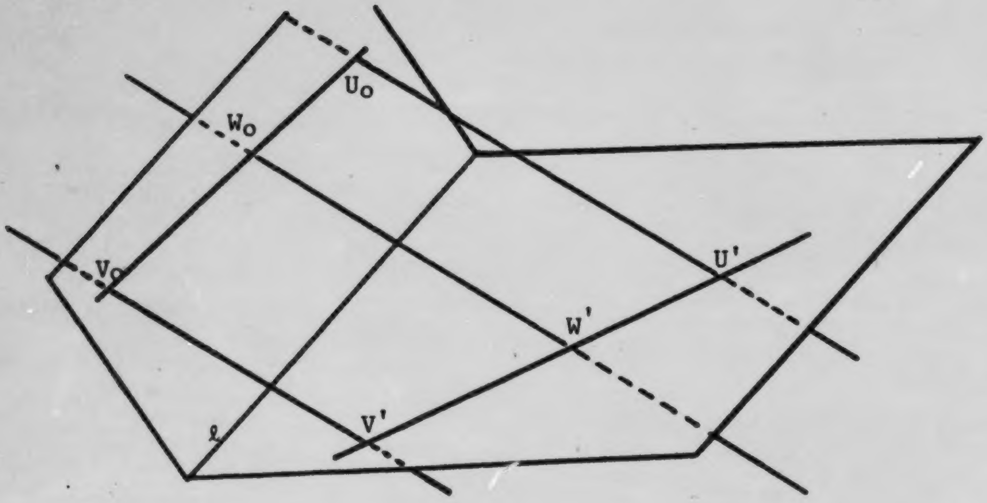


Figure 26

Let U, V, W, X be points in the plane, $\bar{U}$, $\bar{V}$, $\bar{W}$, $\bar{X}$ their surrogates on the acetate which, in the final position, cover the points Uo, Vo, Wo, Xo, respectively, and project into the points U', V', W' and X', respectively, so that U' is the image of the point U, the point V' is the image of the point V, etc. If the lines U V and W X are parallel, then so are their surrogates, $\bar{U}$ $\bar{V}$ and $\bar{W}$ $\bar{X}$, and hence, so are the lines covered by them in the final position of the acetate; namely Uo Vo and Wo Xo. The planes Uo Vo V' and Wo Xo X' are parallel since the lines Uo U', Vo V', Wo W', Xo X' are all in the fixed direction, and hence parallel. Thus, the lines U' V' and W' X' are also parallel. Hence, parallelism is invariant under an axial affinity.

An axial affinity fixes every point on its axis ℓ. Suppose a point P not on ℓ is fixed; that is, P is mapped onto itself. For any point U on the axis ℓ, the affinity fixes U and hence, maps the line P U onto the line P U, so that any point on this line is mapped onto a point on this line, possibly a different point! (See Figure 27.)

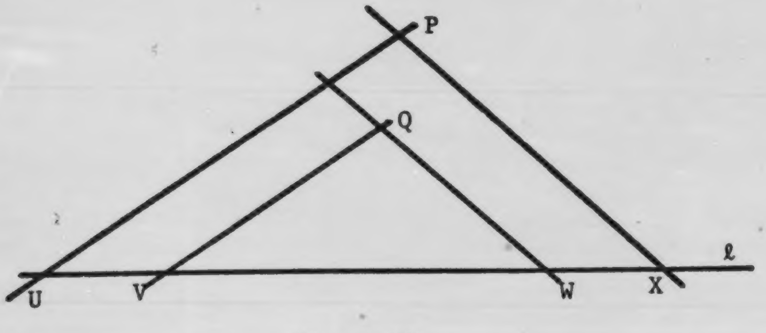


Figure 27

Let Q be any other point in the plane and X a point on the axis different from U. If Q = P or Q is on ℓ, then the image of the point Q, Q' = Q and the point Q is fixed. Otherwise, the line through the point Q parallel to the line P U meets ℓ in a point V. Similarly, the line through the point Q and parallel to P X meets ℓ in the point W. Since the line Q V is parallel to the line P U, and parallelism is invariant, their images under the affinity, P' U' and Q' V', are parallel. However, since P, U, and V are fixed points, Q' V must be parallel to P U. But Q V is parallel to P U so that Q V and Q' V must be the same line. That is Q' is on the line Q V. In exactly the same way, the line P X is fixed so that the line Q W, which is parallel to P X, must be mapped into a line through W and parallel to P X. But there is only one such line, namely Q W, so that Q' must lie on Q W. Thus Q' lies on the line Q V and on the line Q W and hence must be the point of intersection. That is Q' = Q and Q is fixed. We have shown,

> If an axial affinity fixes a point in the plane not on the axis, then it fixes every point. That is, it is the identity.

Henceforth, we assume that the axial affinity is not the identity.

Now let P be any point not on ℓ, and let P' be its image under the affinity. Then P $\neq$ P' and we consider the line P P'. There are two possibilities to deal with. The first of these is that the line P P' is parallel to ℓ (see Figure 28). Since P is mapped into P' and parallel lines are mapped into parallel lines, the line P P' will be mapped into a line through P' (the image of P) parallel to ℓ. But P P' is such a line, so P P' is mapped onto itself. On the other hand, if P P' meets ℓ at U (see Figure 29), then U is fixed under the affinity, which maps the line P U onto the line P' U.

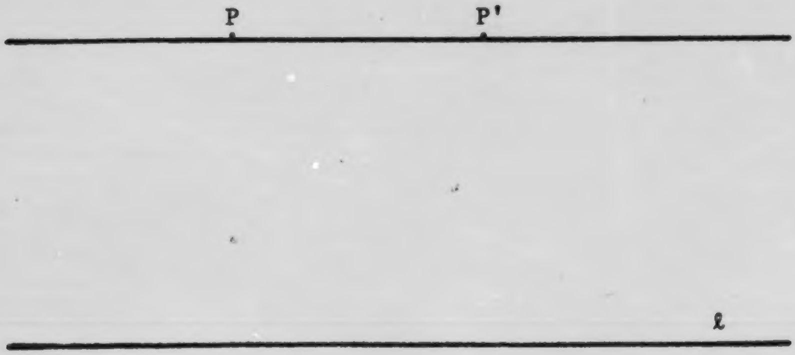


Figure 28

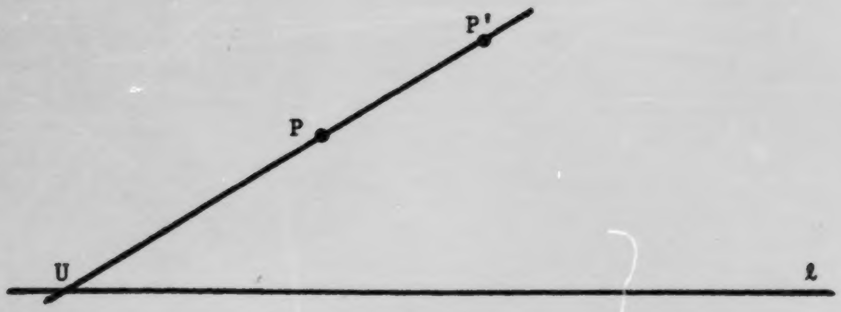


Figure 29

However, since the points P, U, and P' are collinear, these lines are the same, and in this case also, the line P P' is fixed under the affinity. Thus, for any point P in the plane, P not on ℓ, an affinity (not the identity) fixes the line P P'. If Q is another point in the plane not on ℓ, then the line Q Q' is also fixed under the affinity. If the point Q is on the line P P', then, since this line is fixed, Q' is also on this line and the lines P P' and Q Q' are identical. Therefore, consider the point Q not on ℓ and not on P P'.

If the line P P' is parallel to ℓ, then so is Q Q' for any point Q; for if the lines P P' and Q Q' meet in a point X, then the image of the point X under the affinity must lie both on the line P P' and on the line Q Q' since both are fixed. Hence, the image of X must be X itself. But that would give a fixed point not on ℓ (since the line P P' is parallel to ℓ!), which is impossible since the affinity is not the identity. Hence, the line Q Q' is parallel to the line P P' for any point Q not on ℓ.

On the other hand, if the line P P' is not parallel to ℓ, then it meets ℓ in a point X. The line through Q parallel to P P' meets ℓ at a point Y (see Figure 30).

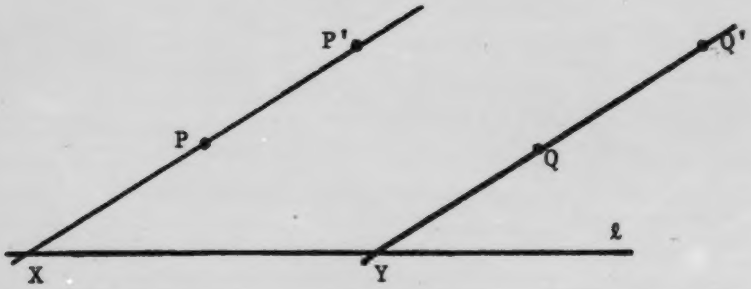


Figure 30

The affinity leaves the line P P' fixed, leaves the point Y fixed, and maps parallel lines into parallel lines. It, therefore, maps the line Q Y into the line Q' Y parallel to the line P P'. Since there is only one line through the point Y parallel to the line P P', Q Y and Q' Y are identical lines and Q Q' is parallel to P P'.

Thus, if the affinity is not the identity, for any points P and Q in the plane, not on ℓ, the lines P P' and Q Q' are fixed and parallel. In other words, an axial affinity fixes all the lines in a given direction (the direction of the line P P' for any point P not on ℓ). Only the identity fixes every line in two different directions!

If this line-wise fixed direction (a direction in which every line is fixed) is parallel to the axis, then the axial affinity is a shear. Otherwise, it is a compression.

Consider a parallelogram U V W X where the point W and X are on the axis ℓ. The affinity maps the line U V onto the line U' V' parallel to ℓ. Moreover, the parallel lines U X and V W are mapped into the parallel lines U' X and Y' W. Thus, the parallelogram U V W X is mapped into the parallelogram U' V' W X (see Figure 31). If the affinity is a shear, then the points U' and V' are on the line U V so that the parallelograms U V W X and U' V' W X have the same base and lie between the same parallels; hence, they have equal areas. On the other hand, if the affinity is a compression then the line U' V', although parallel to the line U V, is nevertheless distinct from it, and hence, in this case the area is changed.

Thus, area is invariant under a shear, but not under a compression.

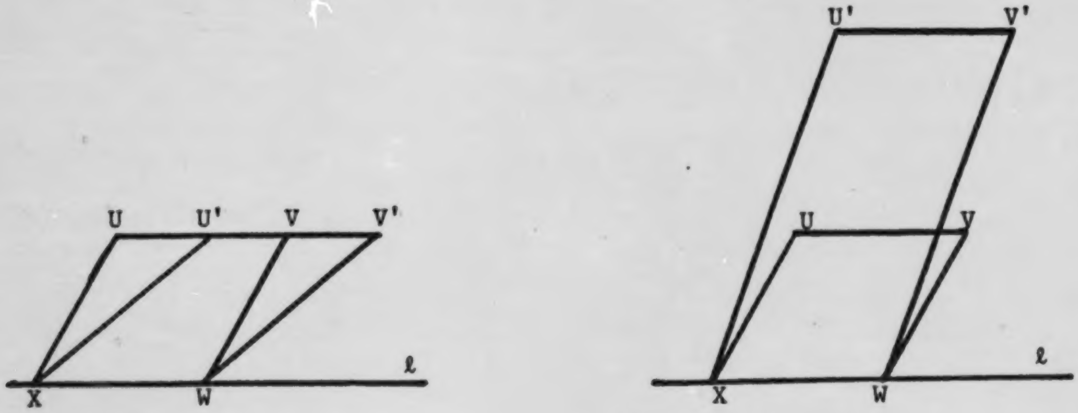


Figure 31

Notice in Figure 31 that the angle $\measuredangle$U X W is mapped into the angle $\measuredangle$U' X W which is certainly not congruent. Thus, the measure of an angle is not invariant under an affinity.

Moreover (see Figure 31), $|U X|$ and $|U' X|$ are in general different, so that length is not invariant under an affinity. However, $|U' V'| = |X W| = |U V|$ since U V W X and U' V' W X are parallelograms! Hence, the length of a line segment parallel to the axis is invariant. Thus, unlike similarities, an affinity has a different affect on the lengths of line segments with different directions.

More specifically, consider the segment $\overline{U V}$. If U V is parallel to ℓ, then, as we have seen, the length of $\overline{U V}$ is invariant. Otherwise, U V meets ℓ at a point W. Let the point $\bar{U}$ on the acetate be the surrogate for the point U, the point $\bar{V}$ for the point V, and the point $\bar{W}$ for the point W. In the initial position, the point $\bar{U}$ covers the point U, the point $\bar{V}$ the point V, and the point $\bar{W}$ the point W. In the final position of the acetate, the point $\bar{U}$ covers the point Uo, the point $\bar{V}$ the point Vo, and the point $\bar{W}$ the same point W. Then the point Uo projects into the point U' the image of U under the affinity and similarly, the point Vo projects into the point V'. Moreover, the lines Uo U' and Vo V' are in the fixed direction and hence, parallel (see Figure 32) so that the triangles Δ W Uo U' and Δ W Vo V' are similar.

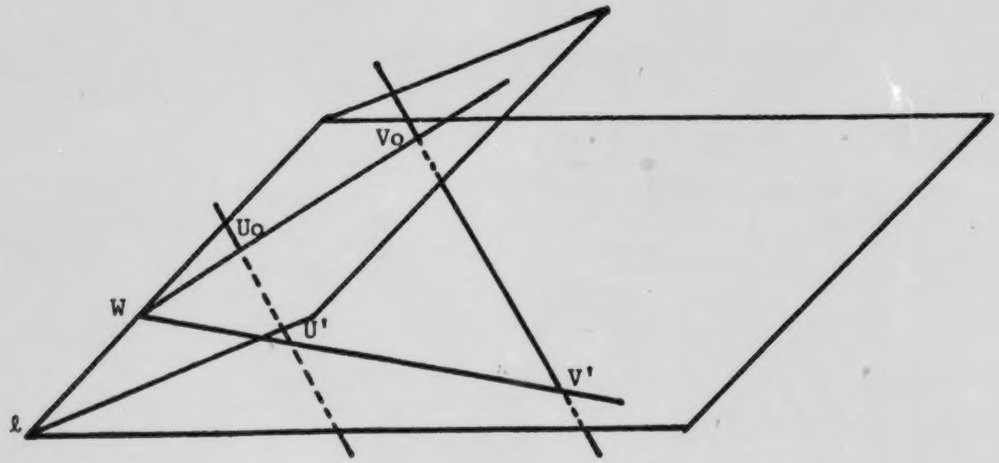


Figure 32

Hence,

$$\frac{|W U'|}{|W Uo|} = \frac{|W V'|}{|W V|} = \frac{|U' V'|}{|U V|}$$

and the magnification of the length of a segment is constant for two segments on the same line. Moreover, if the two segments are parallel, the magnification is again the same; for if, say, $\overline{X Y}$ is a segment parallel to $\overline{U V}$, then the line X Y meets ℓ in a point Z. We choose the points U and V so that U X and V Y are parallel to ℓ. Then the parallelogram U V Y X is mapped into the parallelogram U' V' Y' X' (see Figure 33).

Hence,

$$\frac{|U'\ V'|}{|U\ V|} = \frac{|X'\ Y'|}{|X\ Y|}$$

Thus, the magnification of a segment depends only on the direction.

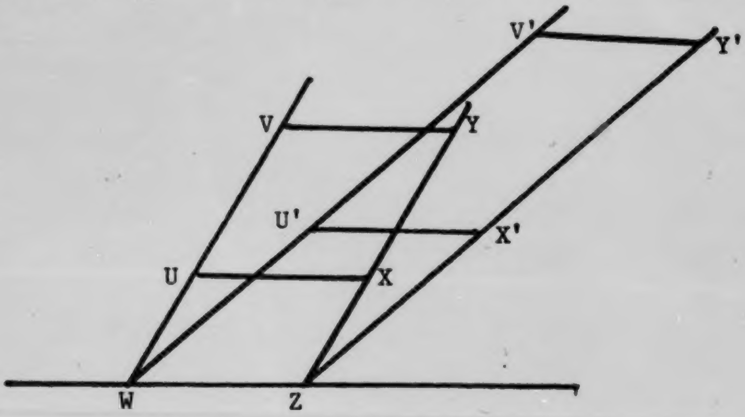


Figure 33

If $\overline{U\ V}$ and $\overline{X\ Y}$ are parallel segments, then

$$\frac{|U'\ V'|}{|U\ V|} = \frac{|X'\ Y'|}{|X\ Y|}$$

so that

$$\frac{|U\ V|}{|X\ Y|} \qquad \frac{|U'\ V'|}{|X'\ Y'|}$$

Thus, the ratio of the lengths of two segments in the same direction is invariant under an affinity.

If U V W X is a rectangle with the line V W parallel to the axis ℓ, then the image of this rectangle under the affinity is a parallelogram U' V' W' Z' such that

$$|V\ W| = |V'\ W'|$$

and

$$\kappa|U\ V| = |U'\ V'|$$

with this magnification factor κ depending only on the direction U V. A simple trigonometric computation yields that the area of the parallelogram U' V' [illegible]' X' is κ' x the area of the rectangle U V W X, where κ' also depends only on the direction U V. It follows from this that any triangle is mapped by the affinity into a triangle with area multiplied by a factor of κ', where κ' is determined by the direction U V (perpendicular to ℓ). Hence κ' is the same for all triangles so that the ratio of the areas of two triangles is invariant under an axial affinity.

Directions, or attitudes, are not invariant under axial affinities but orientation is.

The affine group of the plane is generated by the group of motions and the axial affinities. It can be shown that the group of similarities is a subgroup of the affine group. In addition to the properties listed above which are invariant under the affine group, it can be shown that ellipses map into ellipses (but not necessarily circles into circles), hyperbolae into hyperbolae, and parabolae into parabolae.

Orientation is not an affine invariant since it is not an invariant of the group of motions. However, the subgroup generated by the direct motions and the axial affinities does leave orientation invariant.

Since motions and shears preserve area, the subgroup of the affine group generated by the motions and the shears leaves area invariant. This is the equiaffine group. Similarities in general do not leave area invariant so that the group of similarities is not a subgroup of the equiaffine group.

The Projective Group

The various classes of transformations discussed up to now have all been defined in terms of an initial position of the acetate on the paper and a final position of the acetate which determined the class of transformations. Thus, for motions, this final position was again in the plane of the paper; for similarities, the acetate was parallel to the paper and was projected from a point onto the paper; for affine transformations, the acetate intersected the plane and the projection was parallel. We now study a class of transformations where the acetate again intersects the paper but we project from a point! As we shall see, this creates certain difficulties which will have to be overcome, but we will ignore these for the moment. As before, we focus our attention on the special case where the final position of the acetate results from rotating the acetate about a line ℓ in the plane. This line is then the intersection of the acetate, in the final position, with the plane (see Figure 34). We will refer to the plane in space "covered" by the acetate in the final position, as the final plane, π_0, to distinguish it from the plane of the paper, namely π.

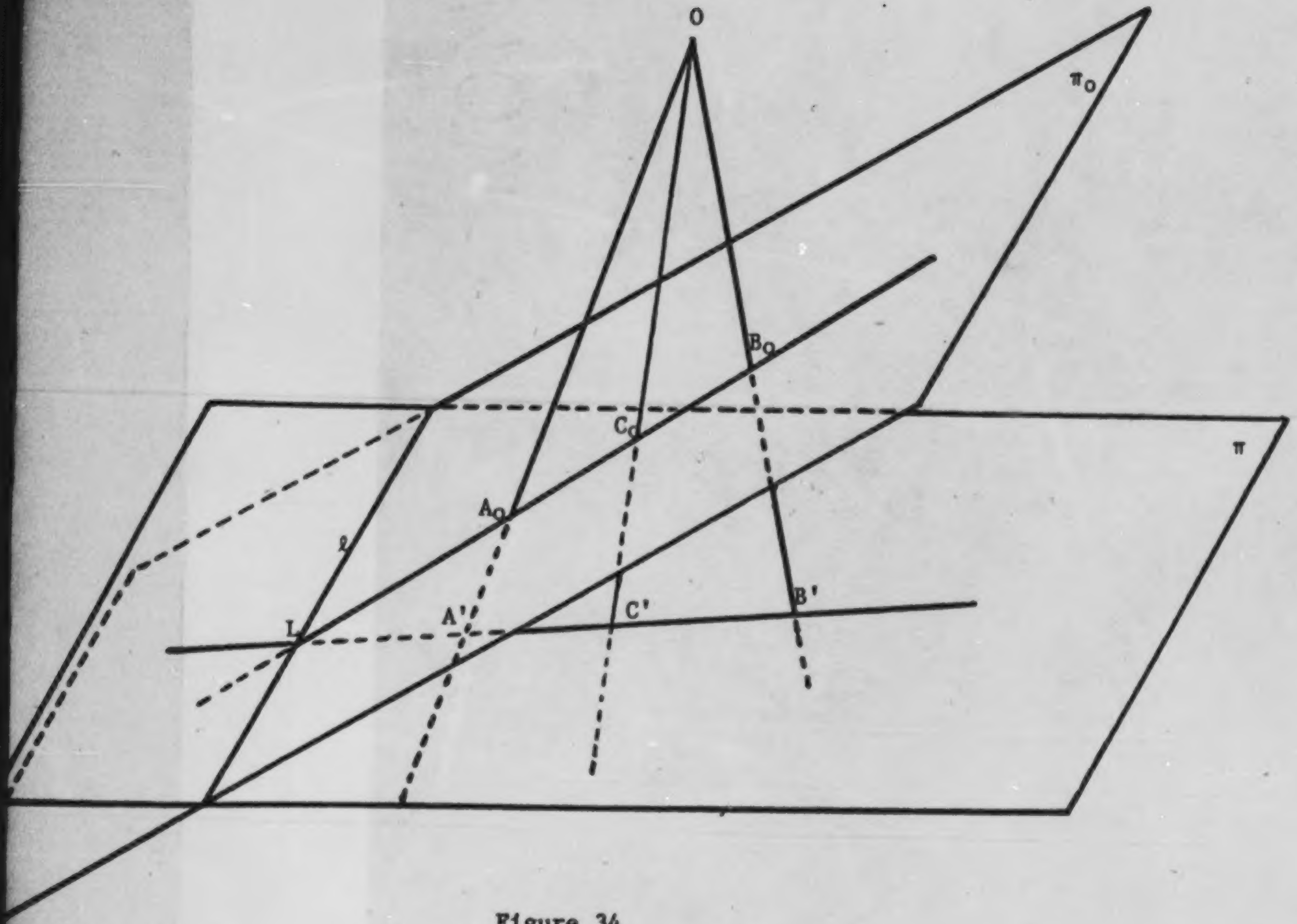

O
π₀
B₀
C₀
A₀
π
ℓ
L
A'
C'
B'

Figure 34

The centre of projection will be a point O not in the plane of the paper or the final plane. A point $\bar{A}$ on the acetate covers a point A on the plane of the paper in the initial position, and a point A_o in the final plane when the acetate is in the final position.

Now consider points A_o, B_o in the final plane (corresponding to points A, B in the plane). The plane OA_oB_o meets the final plane π_o in the line A_oB_o and the plane π in the line A'B'. The line A'B' is then the image of the line AB under this transformation. If C is another point on the line AB, then C_o is a point on the line A_oB_o so that the line OC_o lies in the plane OA_oB_o. The line OC_o then meets the plane π in a point C' which is on the line of intersection of the plane OA_oB_o and the plane π_o—that is, the line A'B'. Thus, when the points A, B and C are collinear, their images A', B' and C' are collinear so that collinearily is invariant under this transformation.

The line A_oB_o meets the line ℓ (where π and π_o intersect) at the point L. This point is on the intersection of the plane OA_oB_o with the final plane π_o and the plane π so that L is also on the line A'B' (see Figure 35).

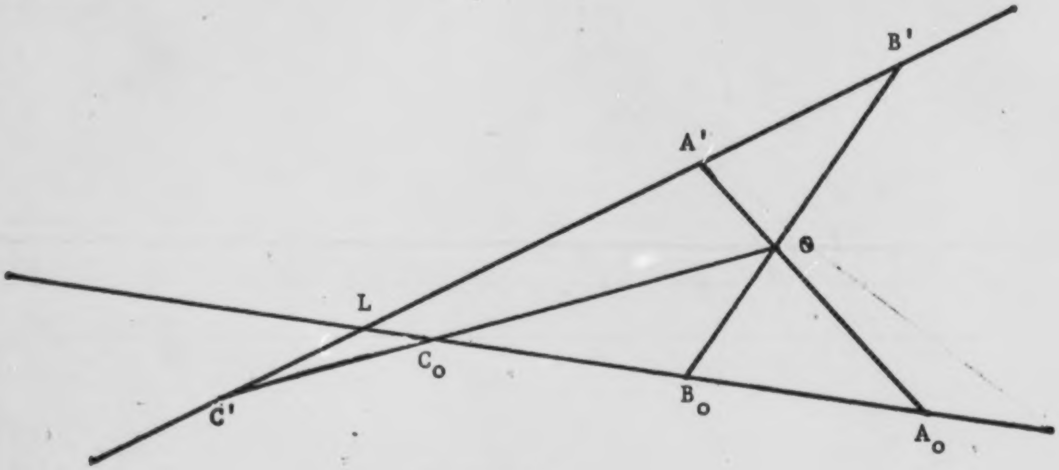


Figure 35

Examining this situation more carefully, in Figure 35, we notice that while the point B_o is between the points A_o and L (corresponding to the point B between the points A and L in the plane) is projected into a point B' not between L and A'. Thus, betweenness is not invariant under this transformation. Indeed, no point between A_o and L is projected into a point between A' and L. Some points in the interval, A_oL, such as the point B_o, are projected in points, such as the point B' on the other side of A' from L. On the other hand, the point C_o also in the interval A_oL is projected into the point C' on the other side of L from A'. The interval $\overline{A_oB_o}$ is not projected into the interval $\overline{A'B'}$ but rather into two pieces, one on the opposite side of the point L from the point A' and one on the opposite side of the point A' from the point L. Thus, line segments are not invariant. Similarly, the ray $\overrightarrow{B_0A_0}$ is not projected into the ray $\overrightarrow{B_0C_0}$ since every point of this ray beyond the point A_0, as for example the point D_0, is projected into a point D' between L and A'. Moreover, the point L not on the ray $\overrightarrow{B_0A_0}$ is projected into the point L on the ray B'A'. Rays are also not invariant under this transformation. Finally, the convex set $\overline{A_0B_0}$ is projected into a non-convex set so that convexity also fails to be invariant under this transformation!

Once again, let L be a point on ℓ the line of intersection of the plane π and the final plane π_0 and let A_0 be a point of the plane π_0 not on ℓ. Consider the line LA in the plane π_0. Let M be another point of the line ℓ and B_0 a point in the plane π_0 such that the line MB_0 is parallel to the line LA_0. The plane OLA_0 and OMB_0 meet in a line m through O (see Figure 36). This line m lies in the plane OLA_0 and if it meets the line LA_0 in the point K say, then this point is on the plane OLA_0, and on the plane π_0, since the line LA_0 is in π_0. But K is also in OMB_0, since the line m lies in the plane OMB_0. Thus, in this case the three planes, π_0, OLA_0, OMB_0 all meet in the point K! However, this means that K is also on the line of intersection of the planes π_0 and OMB_0, namely the line MB_0! It follows from this that the lines LA_0 and MB_0 meet in the point K. Hence, if the lines LA_0 and MB_0 are parallel, they are parallel to the line m which is thus parallel to the plane π_0. That is the line m of intersection of the planes OLA_0 and OMB_0 is parallel to the plane π_0 and hence meets the plane π in some point S'.

The line LA', the projection into the plane π of the line LB_0 is the intersection of π and the plane OLA_0 and hence contains the point S'. Similarly the point S' is on the line of intersection of OMB_0 and π, namely the projection of the line MB_0, the line MB! It follows from this that the parallel lines LA_0 and MB_0 project into lines which intersect (at S')! Hence, parallelism is not preserved under this transformation and hence is not an invariant!

Now consider a point C_0 in the final plane such that the line OC_0 is parallel to the plane π. The plane OLC_0 meets the final plane in the line LC_0 and the plane OMC_0 meets the final plane in the line MC_0 (see Figure 37). The planes OLC_0 and OMC_0 meet in the line OC_0.

The line LC_0 projects into the line of intersection of the plane OLC_0 and the plane π. Let D' be some point on this line. Similarly, the line MC_0 projects into the line of intersection of the plane OMC_0 and the plane π. Let E' be a point on this line. Thus the line LC_0 projects into the line LD' and the line MC_0 into the line ME'. Since the line OC_0 is parallel to the plane π, the lines OC_0 and LD', both in the plane OC_0L, are parallel as are the line OC_0 and ME' both in the plane OC_0M. Thus, the lines LD' and ME' are parallel! However, these lines are the projections of the intersecting lines LC_0 and MC_0. It follows therefore that not only do parallel lines project into intersecting lines but intersecting lines project into parallel lines!!

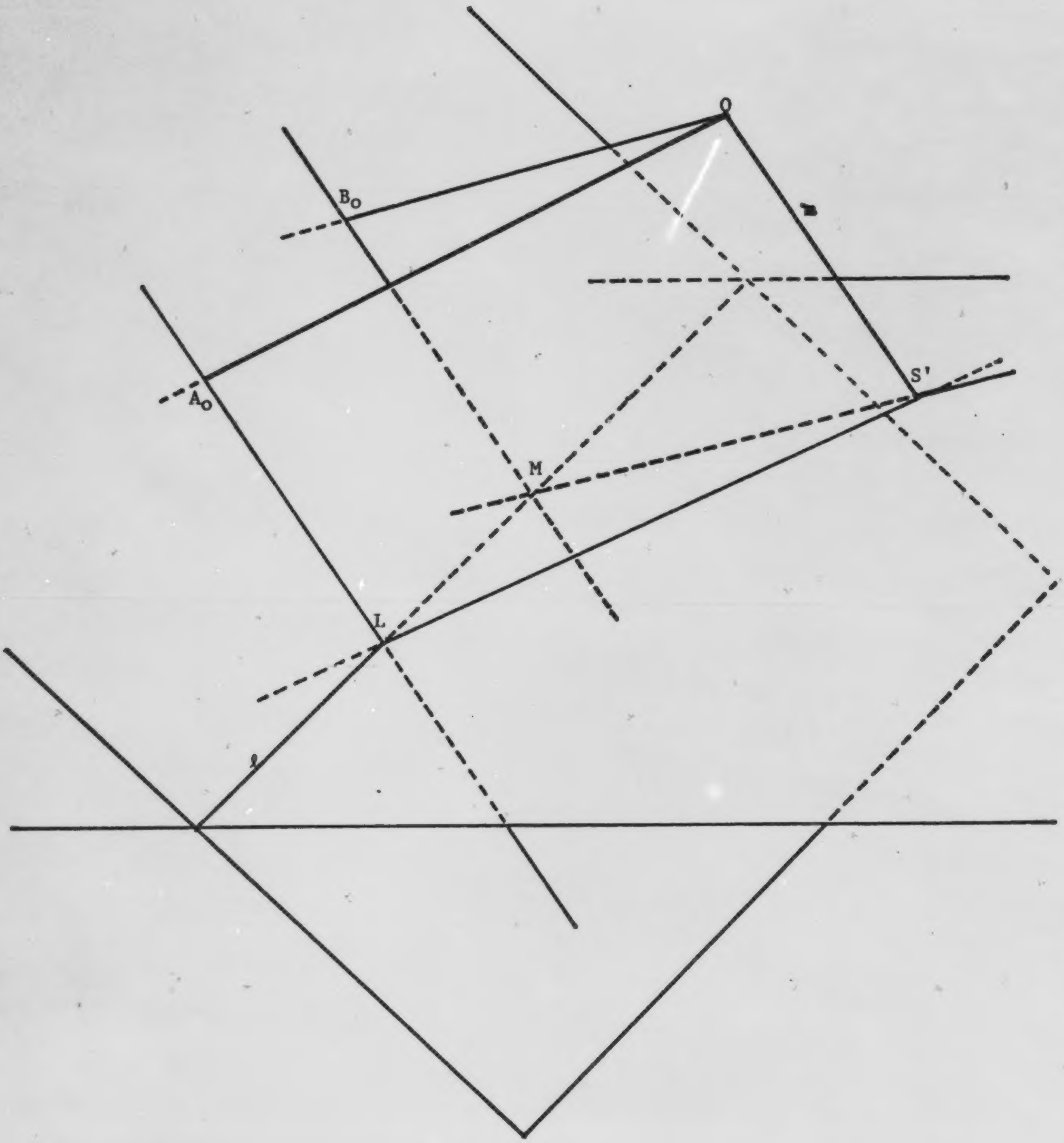

O
B₀
m
A₀
S'
M
L
ℓ

Figure 36

We have here a rather unusual situation. It is one thing when parallel lines project into intersecting lines, for then one might expect that two points, one on each of the lines, have the same image, namely the point of intersection of the projection of the line. While this would mean that what we have defined is not a transformation since condition (ii) for a transformation (see p. 109), is violated that is not the problem here, for it is in fact condition (i) for a transformation which is violated, since the point of intersection S' of the projections of the parallel lines LA_0 and MB_0 is not the image of two points under the projection, but the image of no point of the plane, since the line OS' is parallel to the final plane!

Disconcerting as this situation may be, a more serious problem is created by the projection of intersecting lines into parallel lines, for this means that the point of intersection C_0 (see Figure 37) projects into no point of the plane so that the point C_0 has no image. This violates the definition of a function (see p. 109)! Not only does this "projection" not yield a transformation, it does not even define a function!

The line OC_0 is parallel to the plane π and hence, under projection from the centre O, the point C_0 has no image. Indeed, the plane through the centre O parallel to the plane π meets the final plane π_0 in a line through C_0 and parallel to ℓ and no point on this line has an image under the projection from the centre O! The image of every point on this line is "missing"—the whole line is "lost" under the projection from the centre O.

On the other hand, the line OS' is parallel to the final plane π_0 so that the point S' in the plane π is the image of no point of π_0 under the projection from the centre O (see Figure 36). Similarly, the plane through O parallel to the final plane π_0 meets the plane in a line through S' and parallel to ℓ and no point of this line is the image of a point of the final plane π_0 under the projection from the centre O. Thus, this whole line consists of "extra" or "superfluous" points which are not images of any point of the final plane π_0 under the projection from the centre O.

In effect, there seem to be points "missing" from the plane, the points of the plane π into which the points on the line through C_0 parallel to ℓ project, and the points of π_0 which project into the points on the line through S' parallel to ℓ. The problem is to locate or find these "missing" points!

Consider a point P in the plane. This point gives rise to or determines a line OD through the centre O. This line can be used to "keep track" of the point P much as a shadow can enable us to locate an object we cannot see! Now every point in the plane has such a shadow line, but not every shadow line is the shadow of a point in the plane. The line OC_0 (see Figure 37) is such a line, since it is parallel to the plane π.

NEEDS TO BE
REDUCED

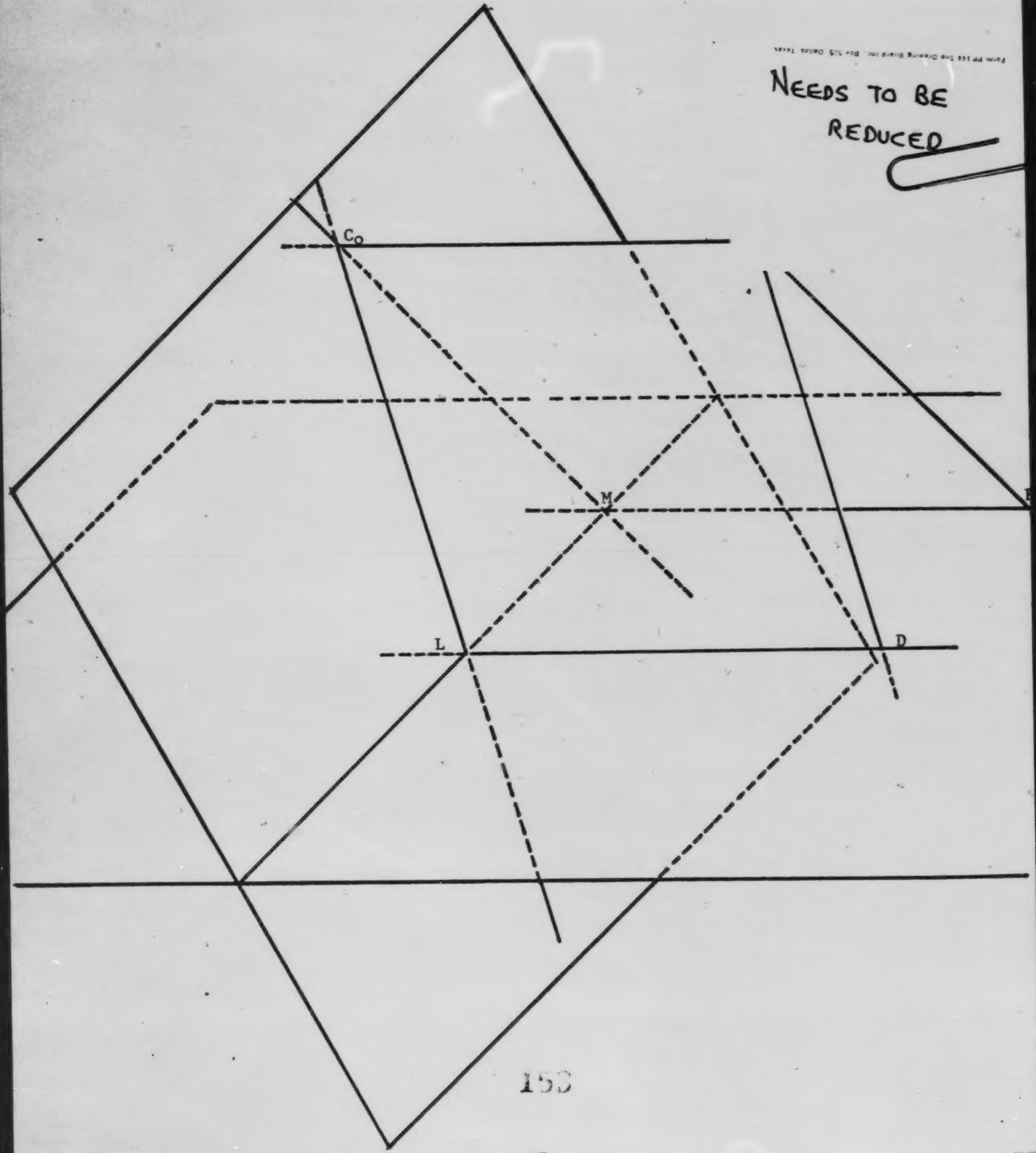


Figure 37

Recall that C_0 had no image under the projection from the centre O precisely for this reason—that the line OC_0 is parallel to the plane π. The "image" of C_0 is one of the "missing" points in the plane. It is not too large a jump to consider this line as the shadow of one of the "missing" points! In other words, we consider each line through the centre O as the shadow of a point of π, and those lines parallel to π are shadows of "missing" points. In this way, we can "locate" the "missing" points by observing their "shadows"! Thus, the projection of the point C_0 is the "point" in the plane π whose shadow is the line OC_0.

Similarly, for the final plane π_0, any line through the centre O and parallel to π_0, as for example the line OS' is the shadow of a "missing" point of the plane π_0, so that S' where this line or shadow meets the plane is then the image of this "missing" point whose shadow is the line OS! Thus, the point whose shadow is the line OS' projects into the point S'.

In this way we have solved the problem raised earlier, for every point in the plane π is an image, but only after we adjoin the "missing" points to the final plane π_0, whose shadows are lines through the centre O parallel to the plane π_0. Similarly every "point" of π_0 has an image —after we have adjoined to π the "missing" points corresponding to shadow lines parallel to the plane π so that the image of the point C_0 of π_0 (see Figure 37) under projection from the centre O, is the "point" of π whose shadow is the line OC_0.

A line PQ in the plane determines a plane OPQ through the centre O which intersects the plane in the line PQ. This plane may be regarded as a shadow of the line! If a point X is on the line PQ then the point X lies in the plane OPQ and hence, so does the line OX. Thus, the fact that the point X lies on the line PQ is reflected in the fact that the shadow of the point X—the line OX lies in the shadow of the line PQ— the plane OPQ!

The shadow of every point in the line PQ lies in the shadow of the line PQ, but the latter contains a shadow which corresponds to no point of the line PQ, the line OZ_0 parallel to the line PQ. Now consider a point X on the line PQ, and its shadow, the line OX. As this line rotates about O in the plane OPQ in a clockwise direction, the point of intersection of this line with the line PQ moves along the line in the direction from the point P to the point Q (see Figure 38). In Figure 38 we have designated several positions of this point of intersection by X_1, X_2, X_3. As the line OX rotates closer to the line OZ_0, the point X of which it is the shadow moves further and further out along the line PQ. In the limit, as the line rotates into the line OZ_0, the point X "goes to infinity." Thus, in some sense, there is a "point at infinity" on the line PQ. This point, whose shadow is the line OZ_0, is one of the "missing" points of the plane! In this same way, there is associated with every line in the plane a "point at infinity" whose shadow lies on the shadow of that line.

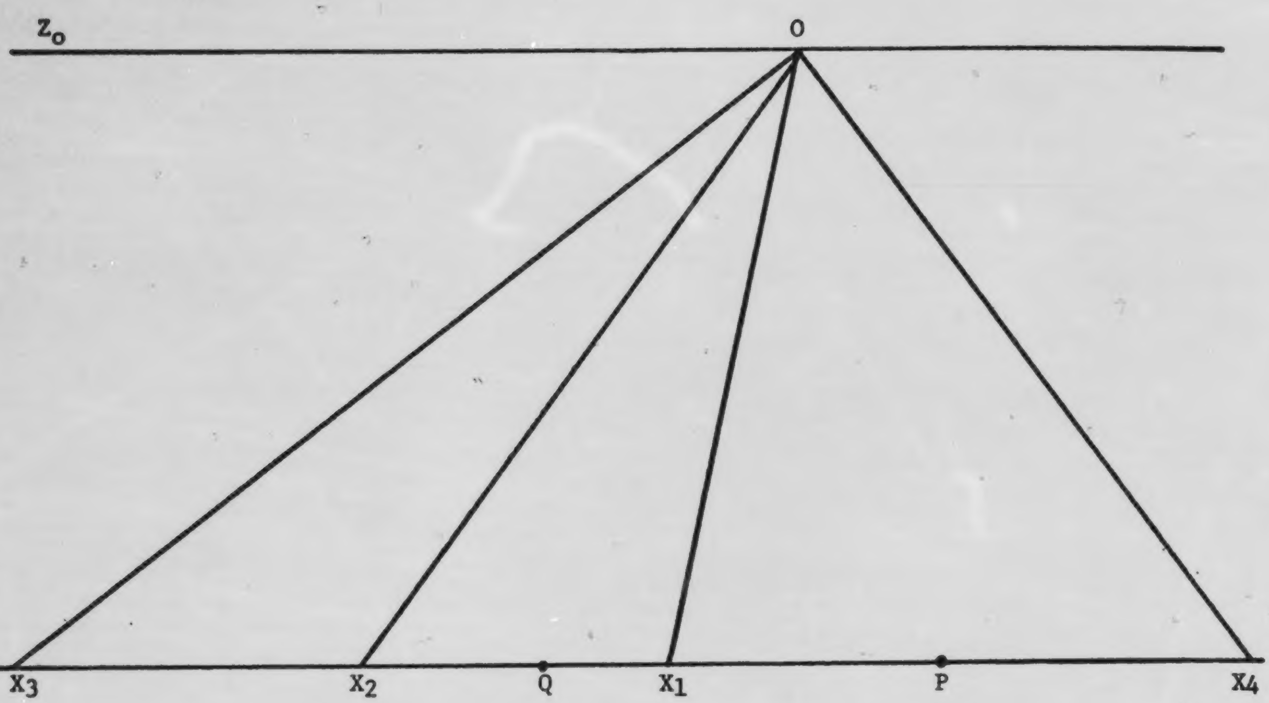


Figure 38

If RS is another line in the plane, parallel to the line PQ then the planes OPQ and ORS intersect in a line through the centre O. As before, this line is parallel to both the lines PQ and RS and hence must be the line OZ_o. Thus, the same "point at infinity" corresponding to the shadow OZ_o lies on the shadow of the line PQ and of the line RS, so that the same point at infinity "lies" on each of the parallel lines PQ and RS. These lines meet therefore "at infinity."

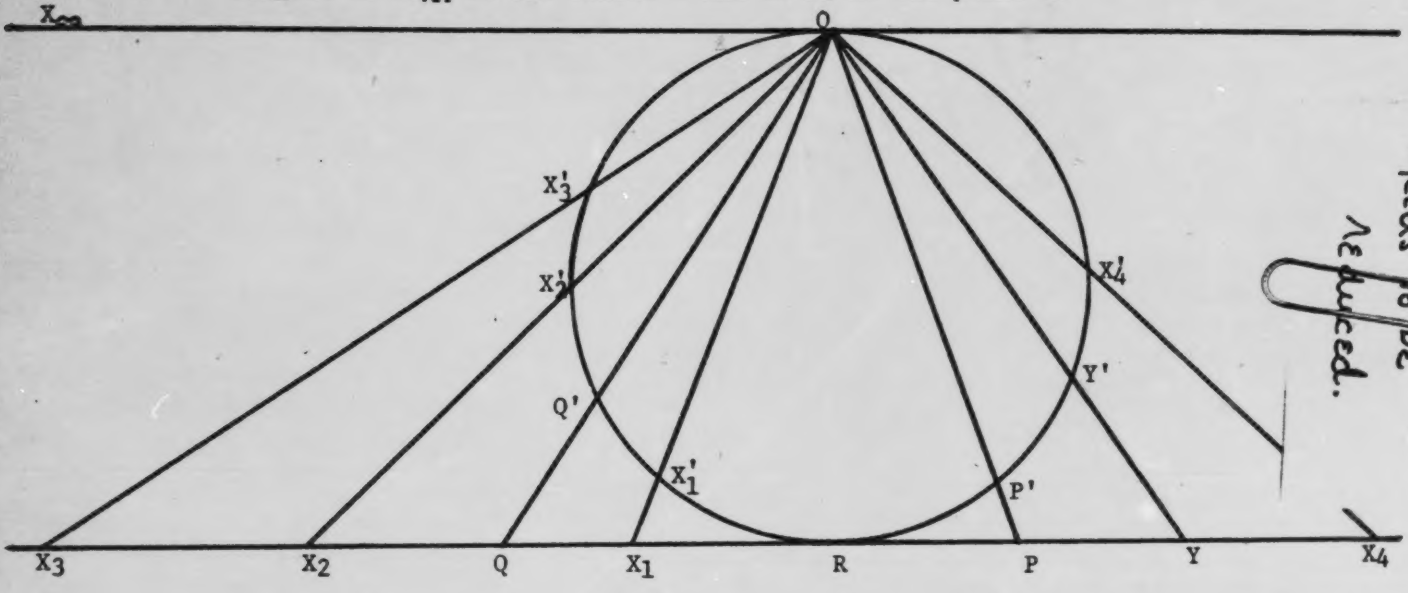
Any line TU in the plane parallel to OZ_o will have as its shadow the plane OTU which also contains the line OZ_o and hence the line OZ_o is the shadow for the "point at infinity" for a whole pencil of parallel lines, the pencil of lines parallel to OZ_o which includes the lines PQ, RS, and TU. Each line in the plane determines a parallel pencil of lines, the set of all lines parallel to the given line in the plane. In turn, there is exactly one line through the centre O parallel to any line in this parallel pencil and this line will be the shadow of the "point of infinity" which "lies" on each line in this parallel pencil. Thus, to each parallel pencil of lines in the plane there corresponds a point "missing" from the plane, the "point at infinity" which is on each of the lines in the pencil!

On the other hand, for any line m' through the centre O and parallel to the plane π, the plane through m' and say perpendicular to the plane, will meet the plane in a line m parallel to m'. This plane will be the shadow of m and will contain m'. Thus, m' is the shadow of the "point at infinity" for the line m'. It follows therefore that each of the "missing" points of the plane π (located by its shadow, a line through the centre O and parallel to the plane π), is the "point at infinity" for some line of the plane, indeed for the whole parallel pencil of lines determined by that line. Moreover, each plane through the centre O, with one exception, is the shadow of a line in the plane π and contains exactly one "point at infinity;" that is, one shadow of a missing point, one line through the centre O parallel to the plane π. The one exception is the plane through the centre O and parallel to π which contains all of these "points at infinity." It is not too surprising that we think of this as a "line at infinity." This is in fact, the missing line of the plane!

Summarizing, we can correct the difficulty in defining a transformation by projecting from the centre O, points in the final position into points in the plane, by adjoining to each plane, the corresponding "line at infinity," as discussed above. In this way, every point in the final plane, regular and adjoined infinite points, has an image under projection from the centre O, a point in the plane π regular or an adjoined infinite point and every point in the plane regular or adjoined infinite point is the image under the projection from the centre O of a point of the final plane, a regular or an adjoined infinite point. In this way, projection from the centre O defines a transformation from the final plane π_0 to the plane π. This in turn yields a transformation from the extended plane π to the extended plane. Henceforth, unless explicitly stated to the contrary, we shall consider the extended plane! This extended plane is the projective plane, and the transformation defined in this way is a projective transformation.

Adjoining the point at infinity, the ideal point as it is sometimes called has a rather interesting consequence. Consider once again the situation in Figure 38, where the point X moves along the line in "the direction from P to Q," that is, the shadow OX of X rotates in a clockwise direction. We have already observed that as the line OX rotates closer to the line OZ_0 the point X moves further out along the line PQ. In the limit as X "goes to infinity" the line OX rotates into the line OZ_0 the shadow of the point at infinity. What happens if the line OX continues to rotate about the centre O in the plane OPQ? The line OX now intersects the line PQ at a point on the other side of P from Q. That is, the point X which went "off to infinity" at "one end" of the line suddenly reappears at the "other end" of the line and as the line OX continues to rotate, the point X moves towards P but still in the "direction from P to Q"! Thus, corresponding to the point X moving along the line PQ from P to Q we have the shadow, the line OX rotating in the plane OPQ about the centre O in a clockwise direction. If the point X starts at the point P (that is the line OX coincides with the line OP), and the point X moves out from P along the line PQ in the direction towards Q (that is, the shadow rotates in the plane OPQ about the centre O in a clockwise direction), then after the shadow has made a half-turn (a rotation through 180^{o}), the line OX will again coincide with the line OP (that is, the point X will have transversed the entire line, including the point at infinity and returned to the point P). The effect then of adjoining the point at infinity is to close up the line so that one can go from P to Q in two ways; one in the direction from P to Q and the other, through the point at infinity by moving in the opposite direction.

This situation is represented more clearly in Figure 39 below where the line OX_{∞} is parallel to the line PQ and the circle is tangent to OX_{∞} at the centre O and to PQ at the point R.



Needs to be reduced.

For any point X on the line PQ, the shadow OX intersects the circle at a point other than the centre, the point we designate by X'. Similarly for any point Y' on the circle different from the centre O, the line OY' meets the line PQ at a point Y. This establishes a 1-1 correspondence between points on the line and points on the circle other than O. As X moves from the point P to the point Q, the shadow line rotates about the centre O in the clockwise direction towards the line OX_{∞} and the point X' moves along the circle towards the point O. The "further out" the point X moves along the line, the smaller the angle between the lines OX and OX_{∞} the closer the point X' moves to the point O. In the limit, as X "goes to infinity" the secant OX moves to coincidence with the tangent OX_{∞} and the point X' on the circle moves to coincidence with the point O on the circle. Thus, the shadow OX_{∞} of the point at infinity on the line PQ corresponds to the point O on the circle. In this way there is established a 1-1 correspondence between points on the extended line and points on the circle.

Corresponding to points P and Q on the line there are points P' and Q' on the circle. We can go from the point P' to the point Q' in two ways; via the point R or via the point O. In the same way, we can go from the point P on the line to the point Q on the line in two ways —via the point R or via the point at infinity. Thus, the point R does not separate the points P and Q on the line because there is an alternate route from the point P to the point Q which does not pass through the point R.

We could define the line segment $\overline{PQ}$ to correspond to the arc P'Q' on the circle which does not contain the point O and this would correspond to the usual notion of line segment, but, as we have already seen, this is not invariant under projective transformations.

Ratios are also not preserved in any form. Indeed, if A, B and C are three defined collinear points and A', B', C' are another three distinct collinear points, then there is a projective transformation which maps A onto A', B onto B' and C onto C'. To see this, place the acetate on the plane for the initial position and let the point $\bar{A}$ on the acetate by the surrogate for the point A, the point $\bar{B}$ for the point B and the point $\bar{C}$ for the point C. In the final position, the point $\bar{A}$ covers the point A' and the acetate is rotated about some line through A' different from the line A'B' through, say, an angle of 45^{o}. Then the point B covers the point B_0 and the point C covers the point C_0 in the final plane and the points A', B_0 and C_0 are collinear. The lines B_0B' and C_0C' meet in a point O in the plane AC_0C' (which may be an ideal point—that is the lines B_0B' and C_0C' are parallel). The projection from O (or parallel projection if O is an ideal point) maps the point B_0 to the point B' and the point C_0 to the point C' so that the transformation defined in this way maps the point A onto the point A', the point B onto the point B' and the point C onto the point C' (see Figure 40).

One may well begin to wonder at this point if there are any projective invariants! Clearly lines are invariant as is the intersection of two lines bearing in mind that a finite point of intersection may be mapped into an infinite point of intersection and parallel lines meet at an infinite point. This leads to an invariant relation among four points of a line which in some sense replaces the notion of a point separating two points.

Consider Figure 41 where PQRS is a quadrilateral, the point A is the point of intersection of the opposite sides PQ and RS and the point B is the point of intersection of the other pair of opposite sides, QR and PS. The diagonals of this quadrilateral, PR and QS meet the line AB in the points C and D, respectively. We describe this situation by saying that points C, D separate the points A, B harmonically. Any projective transformation leaves this configuration invariant so that the images C', D' of the points C, D harmonically separate A', B' the images of the points A, B. Thus, the property of the points C, D of separating the points A and B harmonically, is a projective invariant.

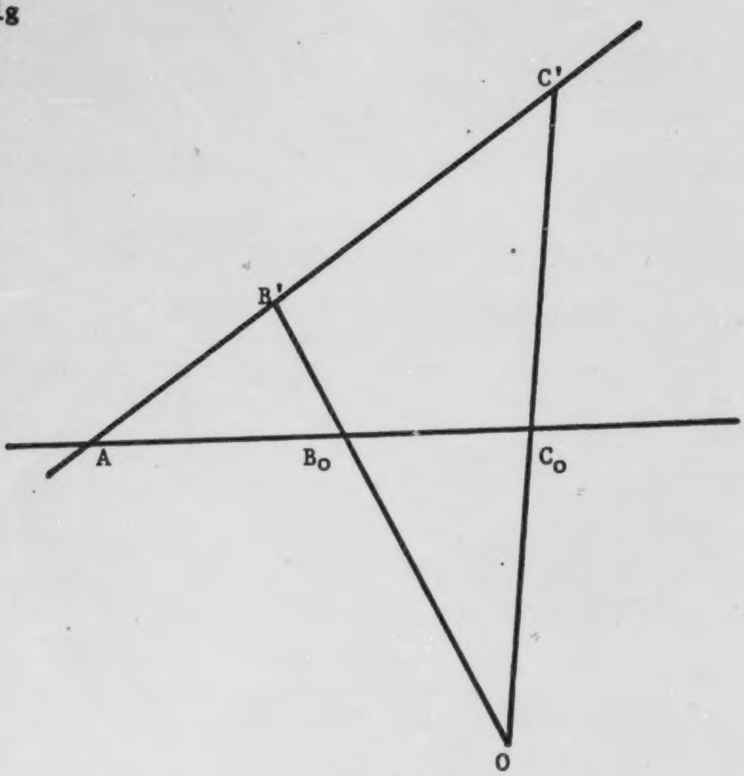


Figure 40

Notice in Figure 41 that if the point D is the point of infinity, then this means that the point C is between the points A and B in the usual sense.

Another projective invariant is the cross ratio of four collinear points, namely

$$\frac{|A \quad C|}{|B \quad C|} \times \frac{|B \quad D|}{|A \quad D|}$$

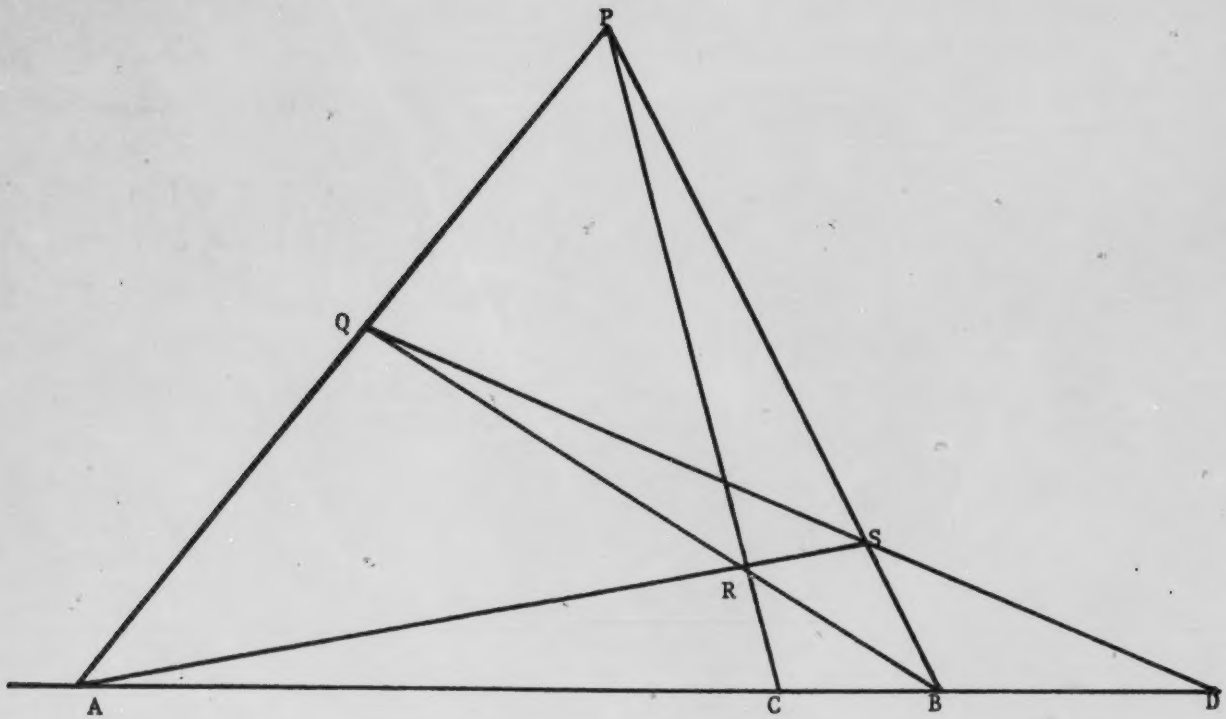


Figure 41

where the points A, B, and C are collinear. Specifically, if the points A', B', C', D' are the projection of the points A, B, C, D (see Figure 42)

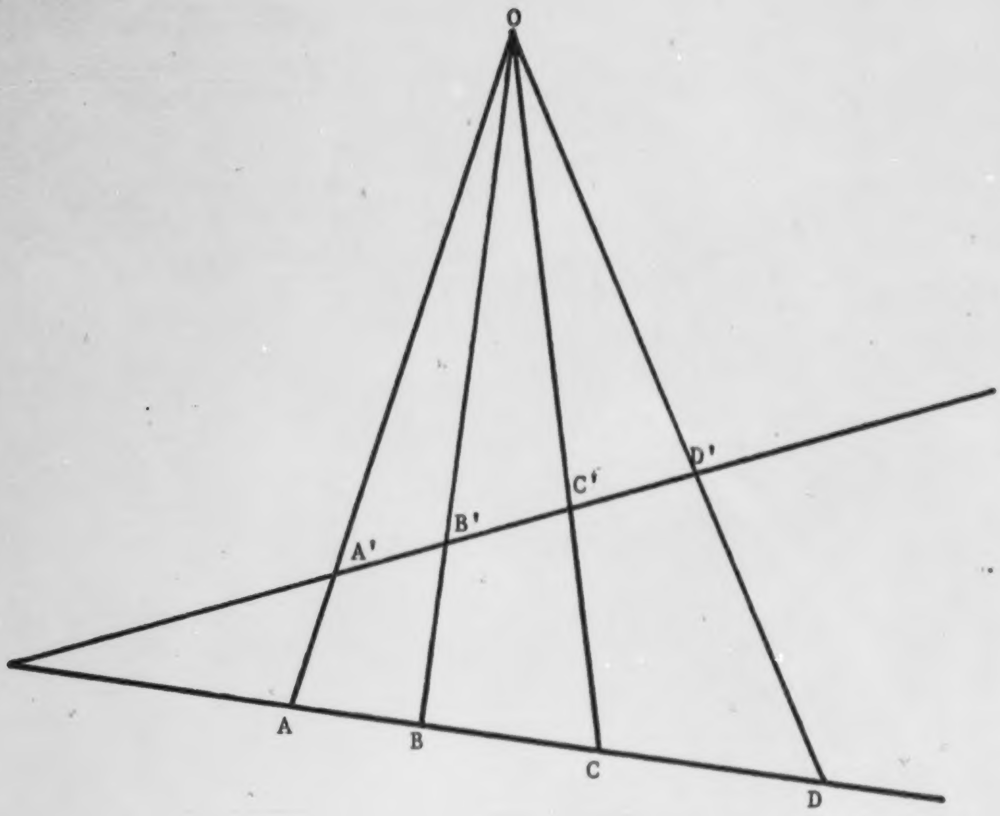


Figure 42

then

$$(*) \qquad \frac{|A\quad C|}{|B\quad C|} \cdot \frac{|B\quad D|}{|A\quad D|} = \frac{|A'\quad C'|}{|B'\quad C'|} \cdot \frac{|B'\quad D'|}{|A'\quad D'|}$$

We may assume that the point O is finite for otherwise the relation (*) would follow from similar triangles. In this case, consider (see Figure 42)

$$\frac{\sin \angle ODA}{\sin \angle OCA} = \frac{\sin \angle ODA}{|O\quad A|} \cdot \frac{|O\quad A|}{\sin \angle OCA}$$

$$= \frac{\sin \angle OAD}{|A\quad D|} \cdot \frac{|A\quad C|}{\sin \angle AOC}$$

Similarly,

$$\frac{\sin < ODA}{\sin < OCA} = \frac{\sin < ODA}{\sin < OCB} = \frac{\sin < ODB}{|OB|} \cdot \frac{|OB|}{\sin < OCB}$$

whence

$$\frac{\sin < AOD}{|AD|} \cdot \frac{|AC|}{\sin < AOC} = \frac{\sin < ODB}{|OB} \cdot \frac{|OB|}{\sin < OCB}$$

$$= \frac{\sin < ODB}{|BD|} \cdot \frac{|BC|}{\sin < BOC}$$

so that

$$\frac{|AC|}{|BC|} \cdot \frac{|BD|}{|AD|} = \frac{\sin < AOC}{\sin < BOC} \cdot \frac{\sin < BOD}{\sin < AOD}$$

Similarly,

$$\frac{|A'C'|}{|B'C'|} \cdot \frac{|B'D'|}{|A'D'|} = \frac{\sin < AOC}{\sin < BOC} \cdot \frac{\sin < BOD}{\sin < AOD}$$

The relation (*) now follows.

Consider now a circle in the plane. This is covered by a circle on the acetate in the initial position. This circle on the acetate covers, in the final position, a circle on the final plane. The "shadow" of this circle, the set of shadows of points on the circle, is a cone with vertex at the centre O (see Figure 43). The projection of this circle will be the intersection of this cone with the plane π, a conic section—a circle (or ellipse) or a parabola or an hyperbola. Which of these it is will depend on the position of the circle relative to the line of intersection of the final plane and the plane through the centre O parallel to the plane π —the vanishing line (which projects into the line at infinity for the plane π).

If the circle in the final plane is disjoint from the vanishing line then the plane π will cut the cone in a circle or an ellipse. If the circle in the final plane π_0 is tangent to the vanishing line then the cone will intersect the plane π in a parabola; finally, if the circle in the final plane intersects the vanishing line—in two points —then the plane π will intersect the cone in an hyperbola. Thus, the projection of a circle is a conic section. More generally, it is not too hard to see that the projection of any conic is again a conic section. It follows, therefore, that conics are projective invariants.

Since the vanishing line projects into the line at infinity, a parabola is a closed curve in the extended plane tangent to the line at infinity. Similarly, the hyperbola which breaks up into two disjoint branches in the Euclidean plane is also a simple closed curve in the projective plane which meets the line at infinity in two points.

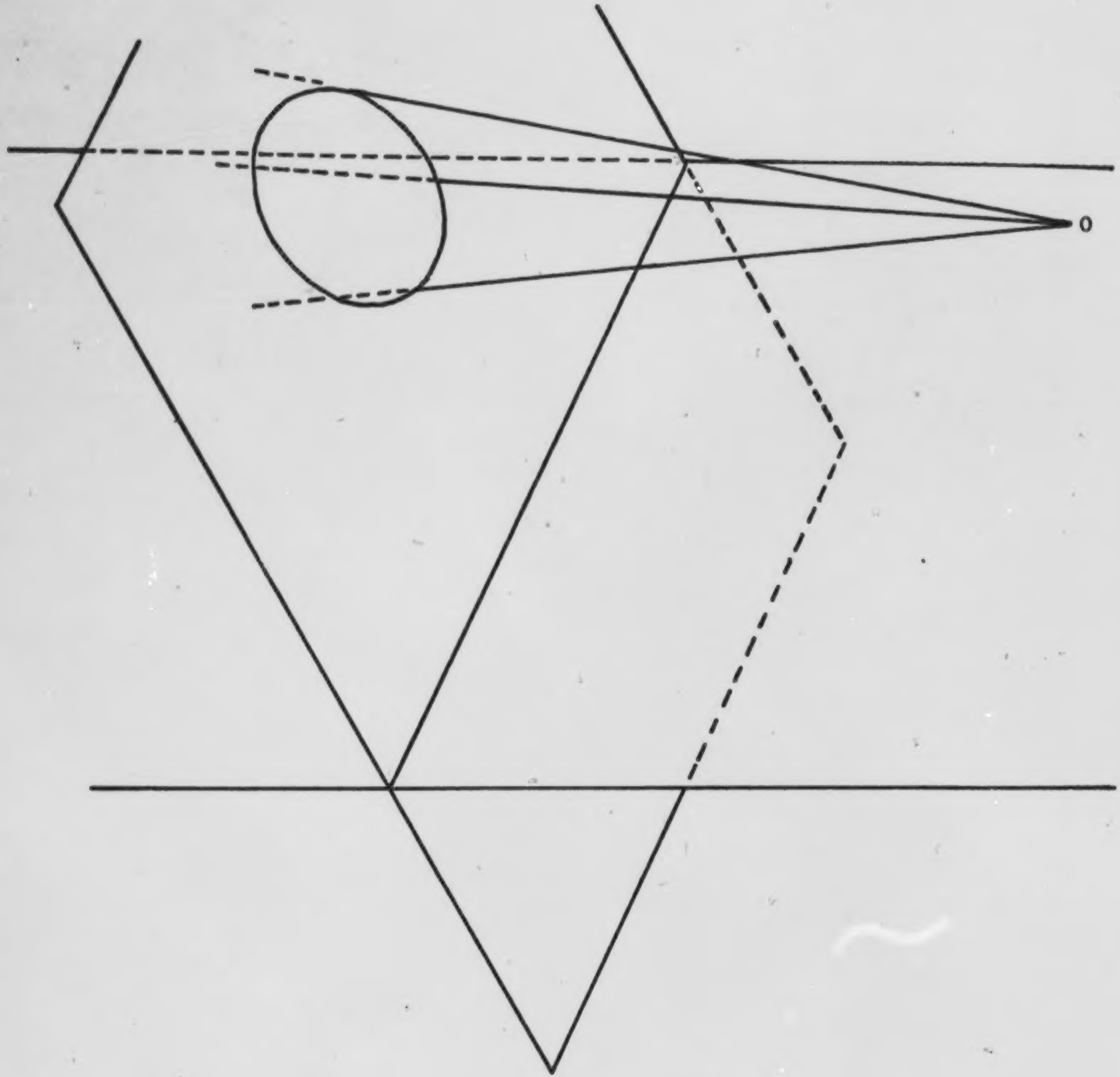


Figure 43

Consider again a circle in the final plane which intersects the vanishing line. The tangents to this circle at the points of intersection meet at a finite point or an infinite point (that is, they are parallel) (see Figure 44).

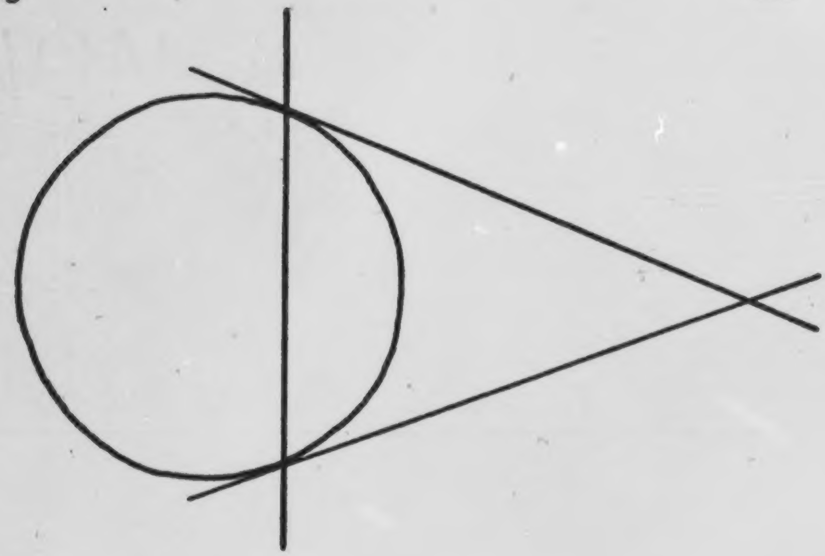


Figure 44

However, the projection of this configuration onto the plane π sends the vanishing line into the line at infinity, the circle into an hyperbola, and the two tangents into the asymptotes to the hyperbola (see Figure 45). If a point moves along the hyperbola from A past B and C and out to infinity then since the asymptote XY is tangent to the hyperbola at a point at infinity, the point at infinity on the hyperbola is the point at infinity on the line XY. Thus, if the point moves past C and past the line at infinity, it appears on the other branch of the hyperbola moving past D through E to F, etc. Thus, going along with hyperbola from B to C and out to infinity or from E past D and out to infinity both result in reaching the same point at infinity, the one on the line XY.

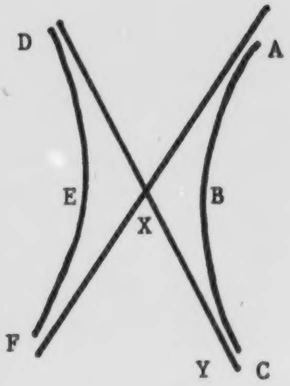


Figure 45

This fact has some very interesting consequences. Consider a line XZ and the hyperbola PQRS (see Figure 46). For a point on the line moving from X to Z, the point P is on the right. Suppose that the point P moves along the hyperbola from P in the direction from P to Q while the point on the line moves along the line in the direction from X to Z. The point on the hyperbola after passing the point at infinity reappears on the other branch moving towards S in the direction from R to S. The point moving on the line, on the other hand, after passing the point at infinity reappears on the line on the other side of X from Z moving towards X. As the point on the line moves towards X, the point on the hyperbola moves towards S. Thus, the point which started out on the hyperbola at the point P on one side of the line ends up at S on the other side of the line, without crossing the lines. For a person standing at X facing Z, the point on the hyperbola starts out at P—on the right—and ends up at X—on the left without crossing over the line XZ! It follows from this that a line does not separate the plane; that is, the projective plane cannot be oriented (see p. 116). The projective plane is said to be non-orientable. In order to get a better understanding of this situation, cut the plane along the hyperbola. Since the hyperbola is projectively equivalent to the circle, this has the same effect as cutting out a disc from the projective plane. What remains—the portion between the two "branches" of the hyperbola containing the line XZ is a Moebius band! Thus, if we take a long strip of paper ABCD (see Figure 47) and join the edges AB and C first giving the strip of paper a twist so that the point C coincides with the point B and the point D with the point A, the point E will coincide with the point F. As most people are now aware, cutting the band along the "line" EF does not result in the band separating into two pieces but it remains in one piece! Thus, in the projective plane not every simple closed curve separates the plane.

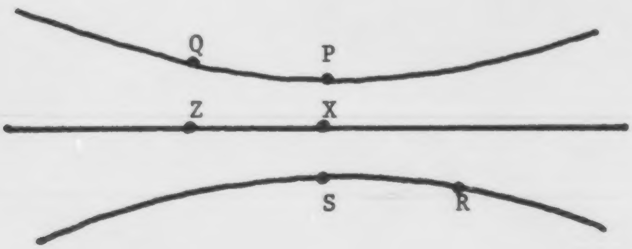


Figure 46

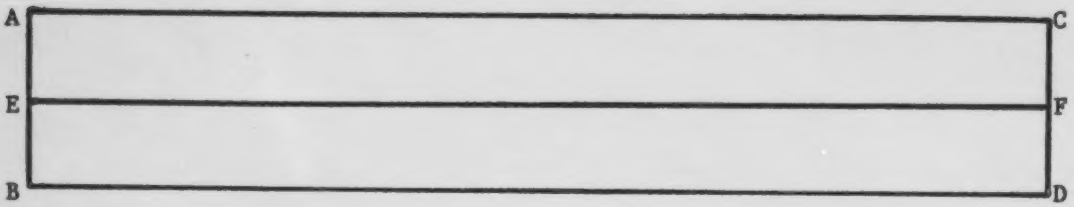


Figure 47

In order to define projective transformations we had to extend the Euclidean plane to the projective plane and the projective transformations were transformations of this projective plane. On the other hand, affine transformations, similarities, motions, etc. were all defined as transformations of the Euclidean plane. However, each of these transformations of the Euclidean plane can be extended to a transformation of the projective plane as follows: Since any one of these transformations, say Q, maps a line into a line, we define the image of the point at infinity on a line under Q to be the point at infinity on the image of the line under Q. Since parallel lines are mapped into parallel lines under Q, this is well defined. The projective group is now generated by the affine group (regarded as a group of transformations of the projective plane and the projective transformations).

The Group of Homomorphisms

The next class of transformations we wish to consider are the homeomorphisms of the projective plane into itself. There are many other classes of transformations "between" the projective transformations and homeomorphisms, such as diffeomorphisms or "smooth" homeomorphisms but these are not usually mentioned in discussions of the Erlanger Program and so we too will omit them.

A homeomorphism is a transformation which does not spread points too wildly. Specifically, if Q is a homeomorphism, P a point, Q(P) = P' then we can assure that Q' = Q(Q) is as close to P' as we require by choosing Q sufficiently close to P. That is, we can "adjust" Q to any desired tolerance; we can adjust the "spread" of the image by choosing points close enough together. More precisely all points which lie within a certain distance of P' lie within a circle of radius that distance (usually called ϵ, the Greek Epsilon). The condition we are describing states that for any such circle C' about P' we can find a small enough circle C about P such that all points within C are mapped by Q into points within C' (see Figure 48).

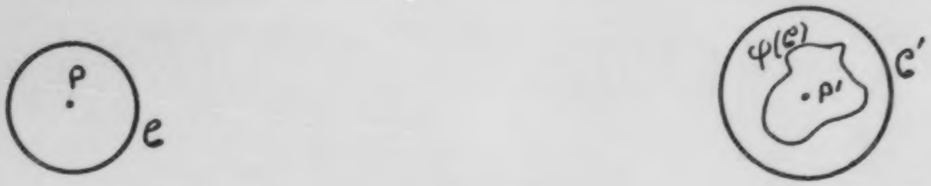


Figure 48

It is in this sense that the spread of points near P is limited. The circle C' about P' determines the acceptable degree of tolerance while the circle C represents the lattitude permitted in order to ensure that degree of tolerance.

What we have actually described is the condition for continuity at P. The transformation Q is continuous if it is continuous at every point in the plane, and a homeomorphism if moreover, Q^{-1} is also continuous. However, due to the special property of the projective plane (it is compact—the Euclidean plane is not!) the continuity of Q^{-1} follows from that of Q.

While our concern here is with homeomorphisms of a particular space—the projective plane—in general, a topological space is a set with the weakest structure which will enable us to discuss continuity. It turns out that for this it is sufficient to require that we be able to specify, about each point, a family of sets of points within various degrees of closeness to the given point. These sets of points are called neighborhoods of the given point and the family of neighborhoods of each point must satisfy certain conditions. Different neighborhoods of a point represent different "degrees of closeness." One way of defining these neighborhoods is by means of a distance function or metric; that is, a neighborhood must contain all points within a certain distance of the given point. This is, in fact, what we have done since it is possible to define such a distance function in the projective plane. Alternatively, the set of points within a given distance of a given point, lie within a circle of radius that distance. A "circle" about a point at infinity is a hyperbola whose axis passes through the point at infinity.

Topology has often been described as "rubber sheet" geometry since "in...topology, figures are not to be considered to be rigid in shape. They may be stretched or squeezed so that they assume different shape." (Schminke, et al., 1973, p. 185) (Copeland, 1974, p. 210).

This is really not a very adequate description. A better description (although still not completely adequate) might be given, again in terms of our acetate sheet where now, before placing the acetate in the final position the acetate may be softened, stretched and compressed in different places and then hardened into its new shape. The important distinction is that stretching a rubber sheet involves a smooth deformation from the unstretched sheet to the stretched sheet, whereas for a transformation it is only the final state which is relevant. Thus, if a movie were taken of this stretching, the transformation would relate only the final frame to the initial frame—all the intervening frames being irrelevant!

Homeomorphisms do not preserve distance or length. Indeed they can distort distances even more than projective transformations. Thus, if the points A, B, C (see Figure 49) are such that the points A and B are close together while B and C are relatively far apart, a homeomorphism can map the point A into a point at infinity and the points B and C very close together. Moreover, even a projective transformation which is a homeomorphism can map the points A, B, C into the points A', B', C' below (see Figure 50).

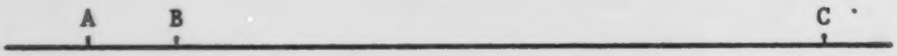


Figure 49

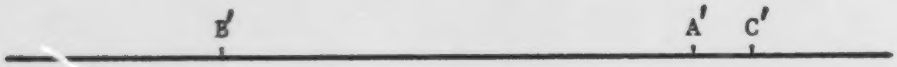


Figure 50

A homeomorphism can distort figures quite drastically so that a homeomorphism of the projective plane can map Figure 51 into Figure 52 and Figure 53 into Figure 54.

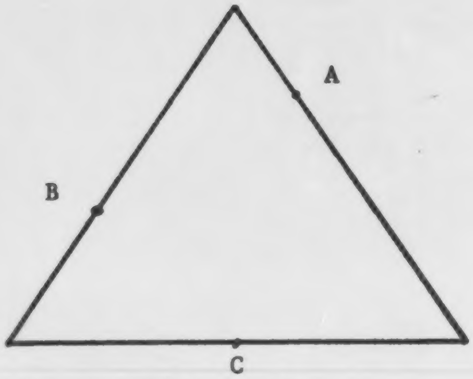


Figure 51

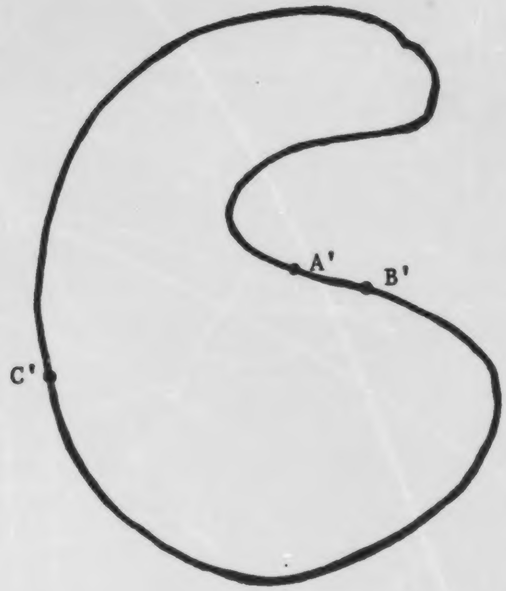


Figure 52

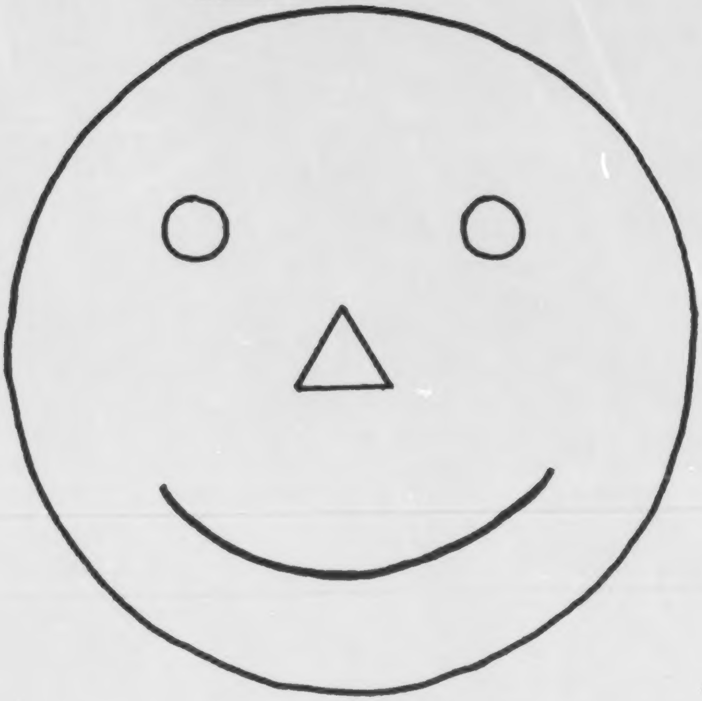


Figure 53

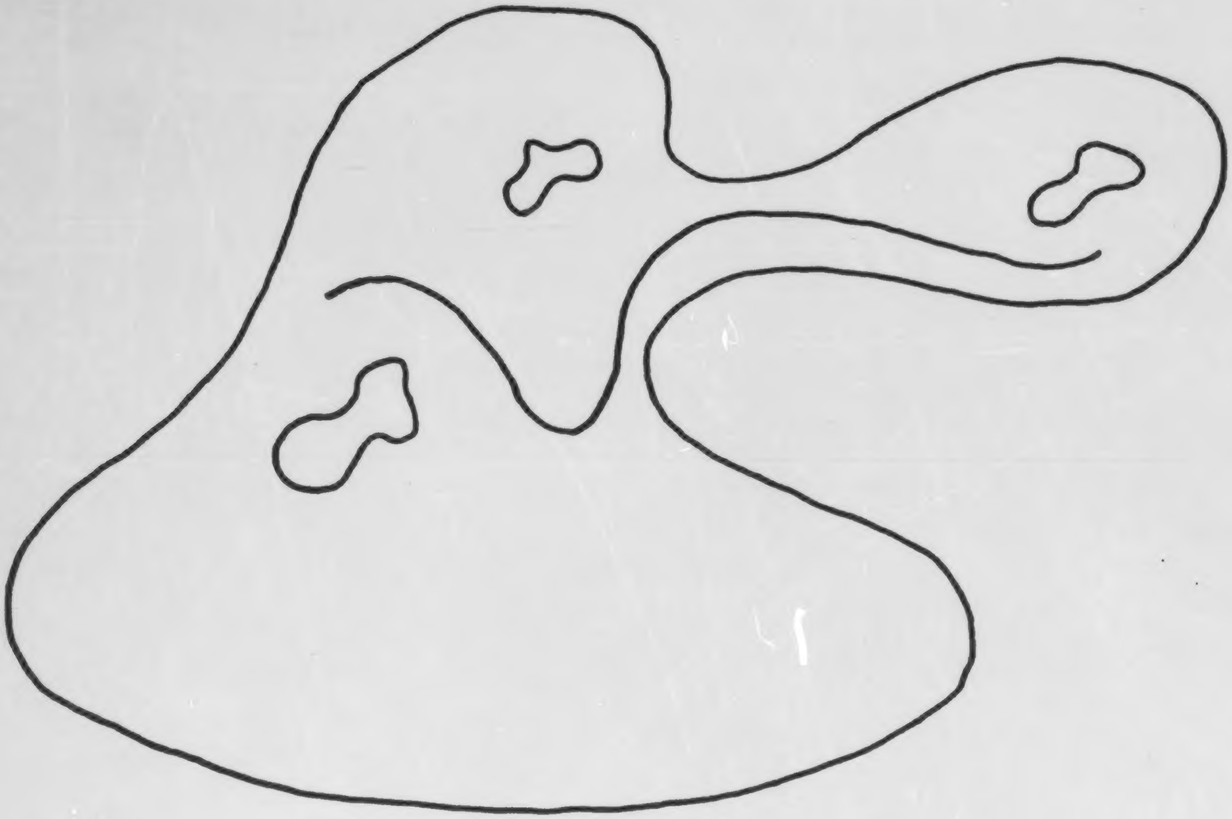


Figure 54

A homeomorphism need not map a line into a line so that collinearily is certainly not a topological invariant. However, the intersection of two lines is mapped onto the intersection of the images of the two lines, which need not be lines.

Lines and conic sections are simple closed curves in the projective plane (see the section on the Projective Group)—that is, they are continuous images of a circle. Indeed any line and any conic section is homeomorphic to a circle. There is still the question of whether it is possible to define a homeomorphism of the projective plane which maps a conic onto a line or conversely. However, a line does not divide the plane whereas a circle, or a conic separates the plane into two disjoint sets each of which it is the boundary (see Figure 55). That is, any neighborhood of any point on the circle contains points from each of these sets. This property of the circle of separating the plane is a topological invariant so that no homeomorphism of the plane will map a circle onto a line!

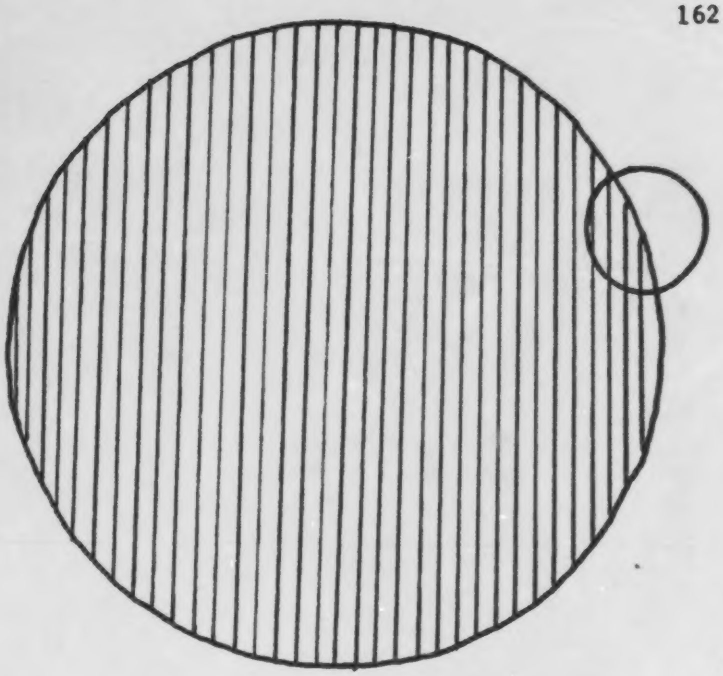


Figure 55

What about the interior of the circle? In Figure 55 it is not too difficult to decide which of the two sets into which the plane is separated by the circle, is the interior. It is clearly the one with no infinite points! But what about the parabola or the hyperbola? Which of the two sets into which the plane is divided should we take as the interior? This question is not so easily answered!

A Consideration of Some Research

We turn next to a brief consideration of the research on the child's conception of space and geometry in the light of our analysis of the Klein Erlanger Program. Among the best known and most widely quoted investigators is Jean Piaget. The most frequently cited result attributed to Piaget is that the child becomes conscious of topological relations before he becomes conscious of Euclidean relations (Copeland, 1974, p. 212).

In *The Child's Conception of Space*, Piaget supplies evidence to support the belief that the child's first concepts of space are topological (Schminke, et al., 1973, p. 185). Thus, Piaget himself has said topological relations are grasped prior to Euclidean shapes (Piaget and Inhelder, 1967, p. 45).

This assertion of Piaget has been subjected to many and differing interpretations. The fact that it is open to so many interpretations serves to indicate a lack of clear understanding of the meaning of this statement. What precisely did Piaget intend by this? What does it mean to say that a child's first concepts of space are topological? One reasonable interpretation might be the following: The child first becomes aware of topological properties of figures and objects before becoming aware of projective and Euclidean properties. That is, the child will distinguish objects which are topologically different—different with respect to certain topological properties, but fail to distinguish between objects or figures which are topologically equivalent but different with respect to projective or Euclidean properties. This seems to accord with what Piaget describes

> ...however, one can begin to speak of real drawings, though curiously enough it is only topological relationships which are indicated with any degree of accuracy, Euclidean relationships being completely ignored. Thus, the circle is drawn as an irregular closed curve, while squares and triangles are not distinguished from circles. (Piaget and Inhelder, 1967, p. 55).

From this it seems reasonable to conclude that when Piaget here speaks of "topological relationships," he is referring to topological properties! Difficulties arise if one tries to follow through with this in a consistent manner. Thus, for example, a child at this stage (3:6 - 4 years) should not be able to distinguish between, say, a sheet of paper, a plate, or a glass, nor between a bead, a washer, a doughnut, and a coffee cup. Similarly, the child should be able to distinguish between a spherical bead (with a hole along a diameter) and a sphere of the same material, size and colour but not be able to distinguish between this bead and a doughnut or coffee cup or between the sphere and a glass or a saucer. Experience with young children at this age shows that they can distinguish between a plate and a glass or between a doughnut and a cup!

At another point, Piaget says:

> our study of drawing and haptic perception showed that the simplest topological relationships such as proximity and separation are also the first to emerge in the course of psychological development (Piaget and Inhelder, 1967, p. 80).

Piaget lists the "most elementary spatial relationship(s)" as: proximity, separation, order, enclosure, and continuity. He goes on to add:

> ...these equally elementary spatial relationships...are none other than those relations which geometricians tell us are of a primitive character, forming that part of geometry called topology (Piaget and Inhelder, 1967, p. 80).

So it is clear that he regards these as topological properties. Moreover, it is also clear from various statements that Piaget is thinking in terms of the Erlanger Program. For example,

> this order of appearance (in the course of psychological development) is also mentioned when space is treated axiomatically by geometricians (Piaget and Inhelder, 1967, p. 80)

(that is, topology emerges firstly, then projective geometry and then Euclidean geometry). In this case, as we have seen, in order for projective geometry to emerge from topology and Euclidean geometry from projective geometry, we must consider as the ambient space, the space in which the objects and figures occur, the extended Euclidean space, the projective space, and it is in the light of this that we must consider the primitive topological relations listed by Piaget.

Proximity is regarded by Piaget as some kind of "nearbyness." How do you specify such proximity? Can it be specified in absolute terms or only in relative terms? How can you compare the proximity of C to B to that of A to B (see Figure 56)?

Figure 56

It seems reasonably clear that Piaget would regard C as proximate to B but not A. Yet, under a homeomorphism of the plane, the point A may be mapped on the point A', the point B on the point B', and the point C onto the point C' of Figure 57, where A' is proximate to B' but C' is not! The point is that no matter how close to the point B we choose the point C nor how far from the point B we choose the point A, under a homeomorphism of the plane we may map A to a point as close as we wish to the image of B and the point C to a point as close to the image of B as we wish. Thus, proximity, as Piaget seems to employ the term, is not a topological invariant.

Figure 57

> A second elementary spatial relationship is that of separation. Two neighboring elements may be partly blended and confused. To introduce between them the relationship of separation has the effect of dissociating or at least providing the means of dissociating them (Piaget and Inhelder, 1967, p. 7).

The nature of the ability to separate points is a property of topological spaces. These are usually given in terms of separation axioms. Thus, in a Hausdorff space, given two distinct points, one can always "separate them;" that is, there are two neighborhoods—one for each of the points which are disjoint. In the space of our experience, if the two points are distinct, then there is a positive distance between them. Balls about each of the points of radius one-third the distance between them (see Figure 58), are disjoint neighborhoods of each of the points.

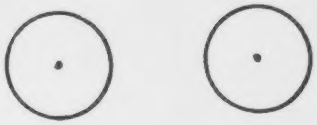


Figure 58

However, it is not exactly clear that this is exactly what Piaget had in mind. Indeed, it seems that what he means is something more complex and more dynamic, as in the following:

> ...as a baby when it sees some object leaning against the wall, appearing as a patch scarcely distinct from its surroundings, there is proximity without clear separation. The more analytic perception becomes, the more marked is the relationship of separation (Piaget and Inhelder, 1967, p. 7).

What seems to come through here is that "separation" as used by Piaget seems to mean the recognition by the child of the possibility of separating two objects in contact. Two balls in contact can be separated (see Figure 59). Yet, when they are in contact their point

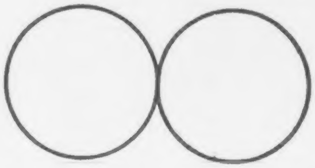


Figure 59

of contact is a single point which would be the intersection of two sets of points. The physical separation of these balls can be described in terms of two paths emanating from this common point of contact. This is not, in the usual meaning, a topological property of this configuration since any homeomorphism of the space will preserve the contact between the balls!

> A third essential relationship is established when two neighboring though separate elements are ranged one before another. This is the relation of order (Piaget and Inhelder, 1967, p. 7).

Later Piaget amplifies on this.

> In the case of a linear series, the relationship of proximity subsisting between separate elements A, B, C, ... etc., is sufficient to provide a basis for the relation of order. And this may be perceived intuitively at an equally early stage of development. The notion of order or sequence is thus a third basic topological relationship, and it is perhaps better to study its psychological development before that of enclosure. Since the relation of between linking an enclosure with a dimension is itself a relationship of order. For example, in the series ABC, B is "between" A and C (Piaget and Inhelder, 1967, p. 80).

What does "one before the other" mean? If A and B are two points in the plane, which is before which? (See Figure 60.)

Figure 60

An example might clarify the situation. If two men are on a ladder leaning against a building, then if the men are ascending we would say that the man higher up the ladder is ahead of or before the other. However, if they are descending the ladder, the one lower down is before the other! Thus, in order to talk about one point being before the other, we usually need a line (or some path) through the two points and a direction along the line—an orientation of the line. Then we say A is before B if the direction from B to A along the line is the given direction; that is, moving along the line in the given direction from B we must pass through A.

Similarly, the point B is "between" the points A and C if they are collinear and in going from C to A along the line in the given direction, we must pass through B so that if A is before B and B is before C, then B is between A and C. It seems clear from the bead task and the clothesline task that Piaget is considering linear ordering in terms of an oriented line as we have indicated. Moreover, circular order is defined in relation to linear order as follows: If the circle is "cut" at some point X and "straightened" out to a line segment (see Figure 61), then a direction along the line containing the segment induces a direction around the circle. If we choose points A and B on the circle distinct from X, then these determine points A' and B' on the segment. The direction indicated, induced by the direction from A' to B' on the segment can be described directly as the direction from A to B without passing through X!

One encounters further difficulties when one recalls that in terms of the hierarchy determined by the Erlanger Program, when considering topology, the space in question is the projective space. In order to define direction along a line or an orientation of a line, one usually makes use of the fact that a point separates a line into two opposite rays. However, this is not true for projective lines!

It is beyond the intended scope of this paper to continue this analysis of Piaget's remaining "topological relations," enclosure and continuity, which present special problems. (Thus, for example, a glass encloses no space topologically yet children at an early age have little difficulty distinguishing the "inside" from the "outside"!) It is clear even from this limited analysis that in building up his theory of the child's conception of space, he leans quite heavily on the Erlanger Program and his understanding of the mathematical concepts related thereto. The conception of his experiments, the focus in the interpretation of his observations, what he emphasizes to support the conclusions he draws—these all depend very heavily on his perception of the mathematics involved. It is equally clear from the analysis that Piaget's understanding of this mathematics is very hazy and unclear and his use of terminology is very different from standard usage. This means that readers of Piaget must be extremely cautious in how they understand or interpret his writings! This caution is not often present with regard to much of the Piaget-inspired research. What is sorely needed is a much more careful analysis of the mathematical concepts and a thorough study of the work of Piaget in the light of the mathematics involved. A first step towards this objective has been undertaken by the author.

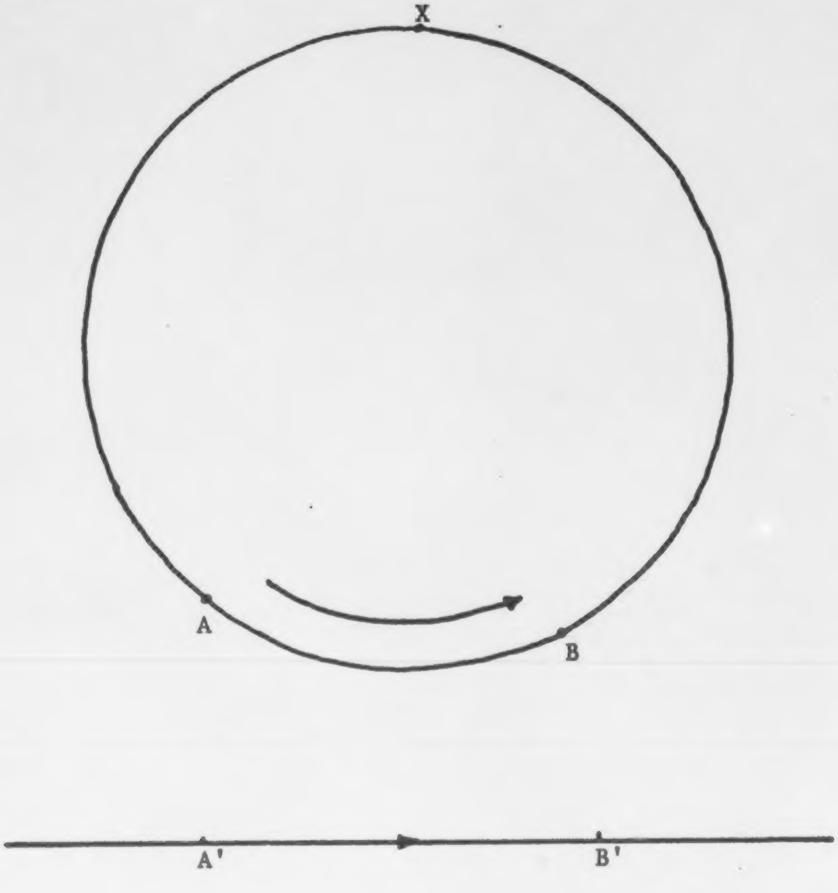


Figure 61

We turn now to a brief consideration of another aspect of research inspired by the Erlanger Program. There has been a great interest in recent years in transformation geometry on every school level—in particular, on motion geometry. As we have already observed, a Euclidean motion other than the identity must be either a reflection, a rotation, a translation, or a glide reflection. This has led to a great focus on reflections, rotations and translations. This has spawned many studies to try to determine at what age a child can grasp the concept of, say, a reflection or whether a child can grasp reflections more easily (or less easily) than say rotations or translations.

A careful examination reveals that they rarely, if ever, deal with transformations. Recall that a transformation of the plane, say, has as its domain, the whole plane; that is, it maps the whole plane onto itself. As such, since distances are preserved, it maintains the relative positions of all points. Thus, a translation other than the identity moves every point in the space so that the image of every point is different from the point itself. This means that sliding an object along the table does not represent a translation for the table itself and the ambient space is stationary! A pencil on the table, or a mark on the table, remain fixed while the object is moved. Moreover, in a translation there are no "intermediate positions" of the object. However, when we physically move an object, we are in fact presented with a continuum of such intermediate positions.

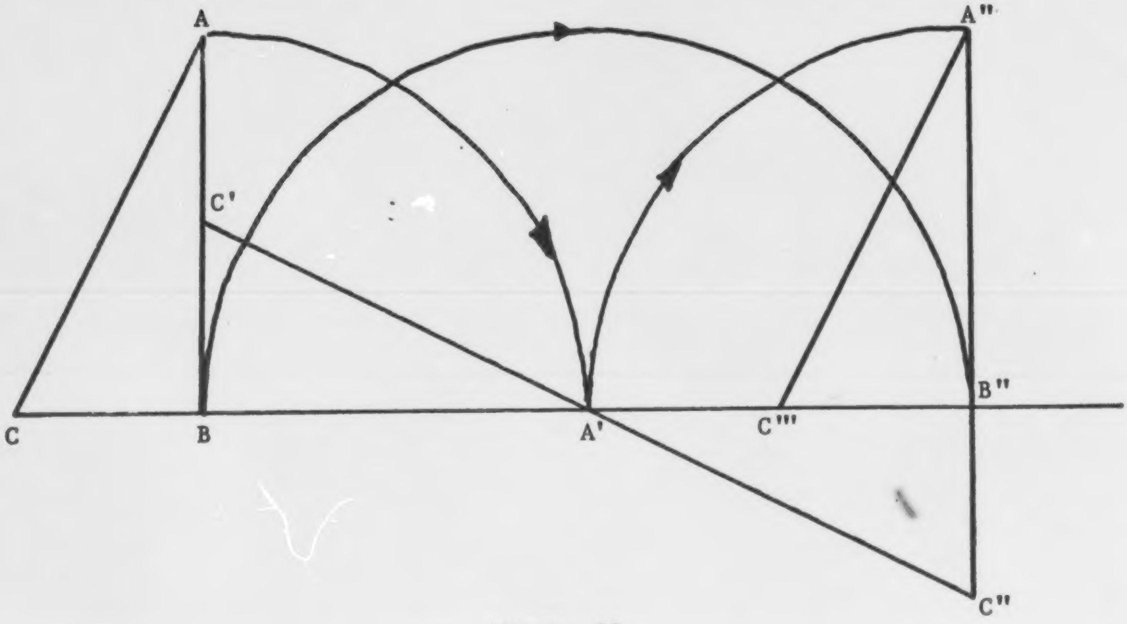


Figure 62

If we start with a card on which has been drawn a triangle Δ ABC, rotate the card through 90° about B, the vertex A traces out the arc of the circle AA'. The card is then rotated about the vertex A (now at the point A') through 180° so that the vertex B traces out the semicircle indicated in Figure 62 to the new position at B". Finally, rotate the card about vertex B (at the point B") through 90° so that the vertex A traces out the arc A'A". Would Δ A"B"C" be considered as a translation of Δ ABC? In fact, Δ A"B"C" is the image of Δ ABC under a translation. Nevertheless, from reading the literature, this action would not normally be regarded as representing a translation—as indeed it does not for the reasons mentioned above (the whole space must be shifted or translated), but sliding the card so that the vertices C and B remain on the line CB" until C coincides with C", B with B", A with A", would be regarded as representing a translation which it does no more than the previous action. What we are dealing with here is not a transformation of the space into itself but rather a continuous, one-parameter family of mappings of a piece of space, the card, into the space which is quite different. This will be discussed more fully in the final part of the paper. The important point that must be confronted is that many studies that purport to deal with transformations do not. They deal with a different concept, a movement or a displacement. It will be necessary to examine much of this work and reinterpret the results in the light of what is actually taking place.

One final point which seems to have been largely overlooked. The fact that any motion can be represented as the product of at most three reflections leads to the classification of motions already mentioned. However, in actual fact, this is an assertion about the structure group, the group of motions rather than about figures in space, or the properties of these figures. As such, studying reflections, translations and rotations is really studying the group of motions rather than the properties of figures. In essence, this might more appropriately be described as group theory—studying the group of transformations—rather than geometry!

A Different View of Geometry

We have already observed that Klein viewed geometry as the study of those properties invariant under a group of transformations. The properties in question are properties of figures—subsets of space. Hence, even in Klein's view, the primary focus of geometry is on the properties of figures.

What then is the role of the transformation group? This serves to delineate the properties of figures. Thus, two figures are essentially the same, or equivalent, if they have the same properties; that is, if one is the image of the other under a transformation of the group. If the group of transformations is the group of motions, then congruent figures are equivalent; under the group of similarities, similar figures are equivalent. The function of the group of transformations is to pick out which figures are equivalent. In fact, the group of transformations partitions the set of figures into equivalence classes. In the case of the group of motions, these are the congruence classes. From the perspective of geometry we are more interested in these equivalence classes than in the group of transformations which effect the partitions of figures into equivalent classes. This leads us to a more general view of geometry.

> A geometry is a set together with a partition of the set of figures (i.e., subsets) into equivalence classes.

A geometry in the sense of Klein is a geometry (or gives rise to a geometry) in this sense where the equivalence classes are determined by the transformation group! This more general definition leads to a notion of geometry more in tune with our experience and provides a more suitable theoretical foundation for the way in which children build up their concepts of space.

However, before expanding on this last point, we call attention to the fact—usually overlooked—that in the Klein view of geometry, it is not so much the group of transformations which determines the geometry, but the geometry which determines the appropriate group of transformations. Thus, we usually start out, not simply with a set and a group of transformations on the set, but a space—a set with structure—and define groups of transformations to be appropriately consistent with this structure. In the case of the plane, we start with lines and a metric or distance function, among other things.

How then do we use this more general view of geometry to select or determine one more in tune with our experiences? We interact with space and objects in space in a dynamic way. Rarely if ever do we experience a transformation of space. The closest thing in our experience to a transformation is to look in a large mirror, for then every point in our side has a reflection in the mirror! Almost invariably, our experience is with a piece of space, a subset of space, a figure or body or object in space which is subjected to displacements or movements in space. Under such a displacement, each point of the figure traces out a path, and we usually observe the figure as it moves from its initial position to its final position. A child handling an object, picking it up and turning it around, is subjecting the object to various displacements. If a moving picture were made of an object under a displacement, for a transformation only the first and last frames would be relevant. However, all the intermediate frames are relevant for the displacement.

The displacements of interest to us here are those which are reversible. (Note that an egg falling to the floor and shattering is not reversible!) Moreover, if a displacement takes a Figure A into a Figure B, and another displacement takes the Figure B into the Figure C, then following the first displacement by the second displacement will take the Figure A into Figure C. We could describe this more precisely in terms of certain function spaces, where an object is a point and a displacement of that object is a path starting at that point. However, it is questionable as to whether this extra precision would add clarity! As an example, consider Figure 63. The pentagon A is slid into B so that each point traces out a line segment. In particular, the vertices A_1 and A_2 trace out segments $\overline{A_1B_1}$ and $\overline{A_2B_2}$ which are parallel and congruent. The second displacement slides the pentagon B into C with vertices B_1, B_2 tracing out congruent and parallel segments $\overline{B_1C_1}$, $\overline{B_2C_2}$. The combined movement takes the pentagon A into C by first sliding to B and then continuing on to C, the vertex A, moving along the polygonal path $\overline{A_1B_1C_1}$.

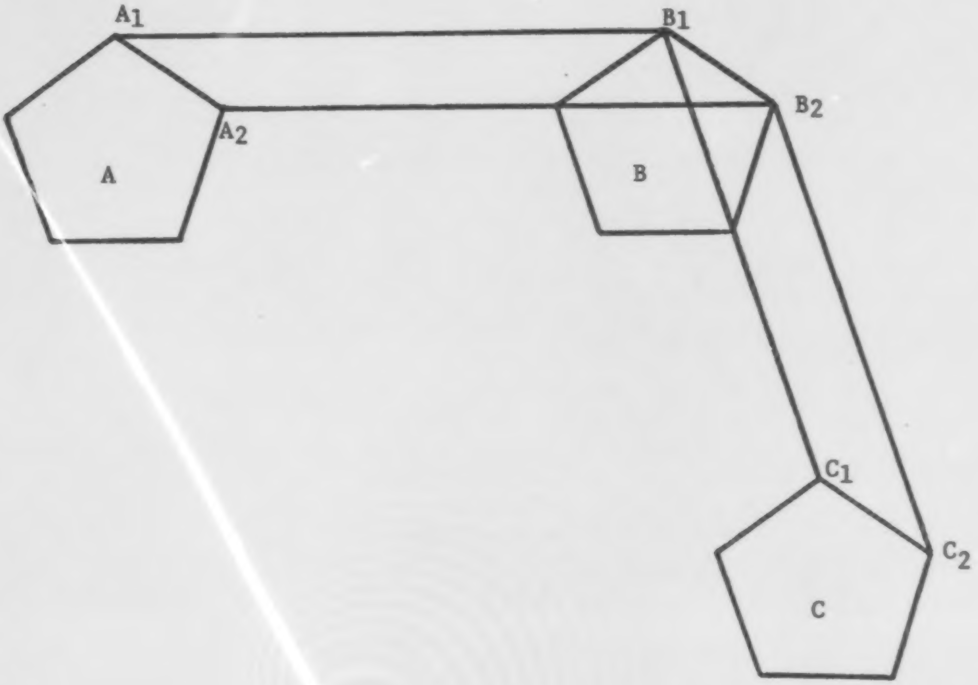


Figure 63

Notice that what we have done in effect is to define a composition of displacements. If we define the pentagon B to be the final figure or image of the displacement and A to be the initial figure then unlike transformations of a space, the composition of two displacements is not always defined. It is only defined when the initial figure of the second displacement is the final figure of the first! In this way we may regard the displacements as morphisms in a category where the figures in the space are the objects! More importantly, we can define an equivalence relation in the set of figures where two figures are equivalent if one is the image of the other under a displacement.

The resulting equivalence classes then define a geometry in our sense. What are the geometric properties of figures in this case? They are properties left invariant by all displacements! As we have already observed, many of the studies purportedly on transformations deal with displacements rather than transformations!

Why are we interested in geometry? What is the relation between geometry and space? We have already remarked that our interaction with space is dynamic. We view objects not from a single perspective but from many perspectives continuously changing. In other words, objects in space are subjected to various displacements! Under the action of a displacement, objects sometimes appear differently and sometimes appear unchanged. Thus, the edge of a glass (see Figure 64) on the table presents the shape of an ellipse. Rotating the glass does not change this appearance. However, tilting the glass, moving it closer (or moving closer to it) or further away does affect the appearance. When the glass is closer, or tilted away from the observer, then the ellipse appears "flatter."

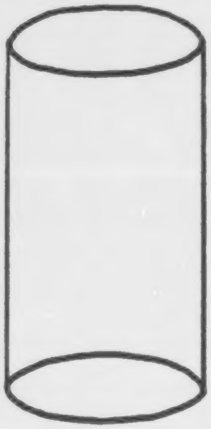


Figure 64

How can we account for the regularities in what we perceive? How can we predict what we will perceive when an object is subjected to a displacement? From our knowledge of geometry and our experience we would say that the edge of the glass is circular in shape and could account for or explain the different appearances of the glass. On the other hand, if the edge of the glass were elliptic in shape, then rotating the glass would change its appearance! (The reader can experiment with a card on which has been drawn a circle and an ellipse.)

Geometry is the structure we impose on space as a framework about which to organize our experiences so that we may account for them, explain them and predict what will happen if....

In constructing geometry we begin with certain primitives. These are usually based on experience but modified in the interests of technical convenience. Points and lines are among the most common primitives in developing geometry but these are the primitives of the mathematician chosen to make his task more manageable. What is a line? How do lines arise? In our experience we never encounter lines which are "infinite" but rather line segments which are finite in length. Technically, it is easier to work with lines than with line segments in developing a geometry. Nevertheless, lines are quite foreign to the experience of children.

A child's first experiences in space are with solid objects, that is three-dimensional figures. Two-dimensional figures are first experienced as surfaces of three dimensional figures. Curves are experienced as edges of solid figures, and points as vertices or corners.

What are the primitives for the child? In subjecting objects to displacements, certain things change; other things remain unchanged or invariant. What is the most invariant figure a child (or adult) experiences? A sphere is certainly one of the most invariant objects for the only thing about a sphere that seems to change under a displacement is its size! A point as the vertex of a solid object is even more invariant! A straight edge, say the edge of a cube, is also extremely invariant under displacements, as is a flat face. It would seem that from the perspective of the child, more natural primitives might be points, line segments* and planar regions or even spheres. While it is possible to develop geometry axiomatically (that is a geometry which reflects our experience and models space) with the sphere as a primitive, it is rather awkward, technically as it is, to work with line segments and plane regions. Rather we make use of the property of a segment that it can always be extended to introduce the notion of a line as a maximally extended segment. Alternatively, collinearity could be taken as a primitive concept, where a set of points is collinear if any three of them lie on some segment, and then take a line to be a maximal collinear set. These are, however, technical devices of interest to the mathematician!

More interesting and more important is the question of how the child structures space. How does the child begin to organize the myriad of experiences into some order? How can we, as educators, facilitate his construction of geometry? Clearly then, if the purpose of geometry is to help the child account for the invariances and lack of invariances under displacements, then he must first become aware of these invariances. How is this to be achieved? This is to be achieved by providing him with many opportunities to displace objects; by presenting him with appropriate experiences which direct or focus his attention on these invariants. For example, the child could be given the task of sorting or classifying in various ways a suitably chosen collection of solid models. In this way, the child's attention is focused on similarities and differences among the solids based on what remains invariant under displacements. He recognizes the special property of the table top which we call "flat" and learns to use it to pick out other flat surfaces. He discovers that a sphere rolls on the table but a cube does not. On the other hand, a cone or cylinder will roll "on its side" but not when resting on one of the flat faces. In this way he begins to develop a notion of "flatness." He begins to acquire a notion of "straightness" in terms of the edge formed by two faces. "Straightness" is thus a common invariant property of various figures. Similarly "flatness" is another property of faces of some solids. When a right cylinder is standing on one of its faces, the other face is "flat." Here the term begins to take on a more special meaning of level or parallel to the table top or to the other face. Further experiences will lead to the refinement of these notions.

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*It is interesting to note the Euclid in his Elements took line segments (which he called lines!) as his primitives.

Figures in the plane often result from drawings or attempts to represent or picture three-dimensional figures. A child must learn to "read" these pictures or diagrams in order to relate these to the appropriate three-dimensional figures or objects. This too requires appropriate focused experiences.

A more detailed and complete elaboration of the ideas briefly discussed above is beyond the intended scope of this paper. One thing, however, should be clear. Through his interactions with objects in space the child gradually becomes aware of certain invariants under movements. He then begins to formulate certain primitive notions with which to build up his geometry. These formulations and notions are not the neat ones of the mathematician but rather messy ones not so easily organized into a tight deductive system.

In order to aid the child in this overwhelming task of structuring his experiences in space, we must understand the problem confronting the child. We must study and analyze very carefully the displacements in the child's experiences in order to ascertain as well as possible those primitives based on which the child begins to structure his experience. Studying these in relation to his experience—in relation to the displacements of his experiences should provide us with insights which will assist us in determining appropriate focused experiences to facilitate the child's understanding of space!

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Students' Understanding of Selected Transformation Geometry Concepts

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Traditionally, transformation geometry has not been included in the geometry courses at the high school level, much less in the junior high or the elementary grades. However, in recent years there have been a number of groups and individuals suggesting that transformation geometry topics be taught in our schools, both at the elementary and at the secondary levels. The Cambridge Conference Committee was one such group, recommending in Goals for School Mathematics (1963) the following topics for grades K-2:

> Symmetry and other transformations leaving geometric figures invariant. The fact that a line or circle can be slid into itself. The symmetries of squares and rectangles...possibly the explicit recognition of the group property in the preceding. (pp. 33-34)

for the 7-12 level:

> ...the motions of Euclidean space are to be treated, leading to the introduction of linear transformations and matrices and the eventual study of linear algebra...the study of geometry could be based on transformations of the plane or of space.... Further study [should take place] to see if an approach could be written up and...[be] appropriate at this level. (p. 47)

and, specifically proposing for the formal geometry course in grade nine:

> Motions in Euclidean space are interpreted as linear nonhomogeneous transformations of the coordinates...treatment of motions in space relative to fixed coordinate axes....Introduce matrices to describe homogeneous linear transformations.... The motions are shown to form a group which is generated by special motions: rotations, translations, and reflections.... (pp. 55-56)

The K-13 Geometry Committee of the Ontario Institute for Studies in Education, in its 1967 report, also listed transformation geometry topics for inclusion in both elementary and secondary grades, and gave a series of reasons for stressing the transformation approach to geometry:

> (1) The main value of motion geometry is in achieving the objective of an informal, intuitive appreciation of geometry. (p. 24)
>
> (2) The study of reflections and rotations provides simple examples of noncommutative systems and may help to remove the anomaly of exposing students only to commutative systems in their study of mathematics. (pp. 21-22)
>
> (3) Motion geometry leads to the ideas of vectors and matrices. (p. 24)

> (4) Motion geometry helps to bring out the significance of the Euclidean concept of congruence, and thus contributes both to new topics (vectors and matrices) and to traditional ones. (p. 25)
>
> (5) Introduction of matrices by means of geometry helps to strengthen the ties which exist between geometry and algebra. (p. 86)

In his article "The Dilemma in Geometry," Carl Allendoerfer (1969) stated that one of the major objectives of geometry in our schools is "an understanding of the basic facts about geometric transformations such as reflections, rotations, and translations" (p. 165). His suggestions for revising our present mathematics curriculum include the teaching of simple geometric transformations to children in the elementary school, and then, at the secondary level, teaching transformation geometry through the use of coordinates and pairs of linear equations (p. 168).

Beyond these recommendations that transformation geometry topics be included throughout the mathematics curriculum, there recently have been several attempts to incorporate transformation geometry into mathematics projects and programs for all grade levels. For example, the University of Wisconsin's Patterns in Arithmetic program included elements of transformation geometry in its television programs for elementary school children. The Developing Mathematical Processes curriculum, produced through the University of Wisconsin's Research and Development Center for Cognitive Learning, included units on rigid motions (flips, spins, and slides), composition of rigid motions, and symmetries in the learning materials for levels five and six (Harvey et al., 1969). Also working at the elementary school level, Marion Walter (1966) designed a unit, Informal Geometry, which used motion geometry to develop the concepts of congruence, symmetry, rigid motion, group, commutativity, and noncommutativity through direct classroom experiences. At the junior high level, a four-book series, Motion Geometry, produced by the University of Illinois Committee on School Mathematics (1967), leads students through a concrete study of slides, flips, turns, congruences, symmetry, similarity, and area. At the senior high level, Coxford and Usiskin (1971) have written a textbook, Geometry: A Transformation Approach, that covers the concepts stressed in the traditional tenth grade geometry course through transformational geometry methods.

While there have been "feasibility studies" that have accompanied many of these curriculum development efforts (Williford, 1972), there is still a need for a more basic type of research concerned with assessing the understandings children have of transformation geometry concepts before they are exposed to any formal instruction in that area. In a paper presented at the National Council of Teachers of Mathematics meeting held at Columbus, Ohio, Steffe (1971) underlined this need:

> Before formal instruction in space and geometry is done, it would be of value to know how children conceive of space and geometry, for such conceptions may drastically affect what children learn about those subjects.

Some sort of base line needs to be established.

There are several areas of student understandings of transformation geometry concepts that should be assessed. A logical place to begin a study of transformation geometry is with three of the basic isometries--rotation, translation, and reflection/flip. Students at all educational levels certainly have had informal experiences with these three transformations. Children look in mirrors and see images reflected there; they watch a record rotate on a turntable or are themselves rotated as they ride on a merry-go-round; they use the concept of translation whenever they slide along a slippery floor or push a toy car in a straight line path. Yet, do students recognize certain properties of these three transformations through their exposure to them on an informal basis? What misconceptions result?

Specifically, do students recognize properties which remain invariant under the transformations of rotation, translation, and reflection/flip? Do they understand the effect of these three transformations on the "sense" and position of figures in a plane? And, what is the relationship between these student understandings and student characteristics such as IQ, age, sex, and Piagetian stage (specifically, conservation of length)? The goals of the present study are to provide some of the base line data on student understandings, to look for patterns and relationships that should be taken into account when transformation geometry curriculum materials are developed, and to identify variables which might be investigated further under more rigorous experimental conditions in future studies.

The investigation reported below is part of a larger study in which a series of nine Piagetian-type tasks were constructed by the investigator and used to assess elementary and secondary school students' understanding of six properties of the basic transformations rotation, translation, and reflection/flip. In the present paper, selected results of student performance on three of these tasks will be discussed. For further details of the complete study, see Thomas (1976).

Research Design

Three sets of investigator-constructed tasks were uses in assessing the following properties* of rotations, translations, and reflection/flips:

(1) Invariance of length--rotations, translations, and reflection/flips each preserve the length of the sides of a geometric figure.

Task set: Shorter-Same-Longer Tasks

Population tested: Students in grades 1, 3, and 6

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* Here, "property" is defined as an effect of a transformation upon a plane figure in terms of the relationship between the original figure and its image.

(2) Invariance of incidence--rotations, translations, and reflection/flips each preserve the incidence of points and lines.

Task set: Before-After Points Tasks

Population tested: Students in grades 1, 3, and 6

(3) Orientation in the plane--rotations, translations, and reflection/flips will each affect the up-down and left-right characteristics of a figure in a unique manner.

Task set: Alphabet Tasks

Population tested: Students in grades 3, 6, 9, and 11

The tasks were given to students in an individual interview setting; each interview was conducted by the investigator.

Students who participated in the study all came from one middle-class socio-economic area of Columbus, Ohio. A total of three schools were used in the study: a senior high school, one of its feeder junior high schools, and an elementary school which fed into that junior high. Ten students (five girls and five boys) were chosen at each of the grades 1, 3, 6, 9, and 11. A stratified sampling technique was used in selecting five average students, two or three above-average students, and two or three below-average students at each grade level.

At each of the three schools a special isolated interviewing room was provided. Students came one at a time to be interviewed by the investigator. Each student sat at a table, and the investigator sat on the same side of the table at the student's right. Paper, pencil, and a clear ruler were provided for the student's use when necessary. Instructions for each task were given orally by the investigator. If a student did not seem to understand the directions or the questions, the investigator rephrased them (or illustrated them, when appropriate) until the investigator was certain that common communication had been established.

Task Descriptions, Testing Procedures, and Results

Shorter-Same-Longer Tasks

When a geometric figure is rotated, translated, or flipped, the length of each of its sides remains invariant. The Shorter-Same-Longer tasks were designed to measure student awareness of this fact. The task set was divided into two series. The first series consisted of showing students one geometric figure (a triangle), then moving that figure under a specified transformation and asking the student to compare the length of a given side to that same side before it had been moved. The second series of tasks used two triangles which the student had chosen and confirmed as matching as to size and shape. Each of these tasks then began with the two congruent triangles superimposed. The top triangle was moved according to a specified transformation, and students were asked to compare the length of a given side of the second triangle in its new position (the "image") to the corresponding side of the first triangle in its original position (the "pre-image").

Since the phrases "shorter than," "longer than," and "the same length as," were going to be used in testing the students, the investigator checked first to determine if the children understood this vocabulary. A series of drawing and selection items was used. All of the students tested got all of the items correct.

Because it seemed as if a student's stage on the standard Piagetian conservation of length tasks might be related to his answers on the Shorter-Same-Longer tasks, Piagetian staggered line tasks were given first to students to determine whether they were conservers or non-conservers in the standard sense. (These tasks are described in detail on pages 95-103 of The Child's Conception of Geometry by Piaget, Inhelder, and Szeminska, 1969.) Results showed that eight first graders (out of the ten that participated in the study) were non-conservers, seven third graders (out of ten) were non-conservers, and no sixth graders were non-conservers.

A total of 30 students took the Shorter-Same-Longer tasks. The KR-20 reliability estimate for this total population was .91.

Table 1 shows means and standard deviations of student scores on five groupings of Shorter-Same-Longer tasks: (1) tasks involving rotation, (2) tasks involving slides, (3) tasks involving flips, (4) tasks which used just one figure (students were to compare the figure in its new position with the same figure before it had been rotated, slid, or flipped), and (5) tasks involving two congruent figures (the two figures were superimposed, then the top figure was rotated or slid). Since not all of the first graders were tested on all of the tasks, the scores recorded in table 1 are the percent of items answered correctly in the specified categories.

Hypothesis 1: Students who are not conservers of length (in the standard Piagetian sense) do not perform differently on the Shorter-Same-Longer tasks than students who are conservers.

A chi square test was run on each of the five groupings of Shorter-Same-Longer tasks. In each case, the chi square contingency table was set up with conservers vs. non-conservers along one dimension, and "number who answered all items correctly" vs. "number who missed at least one item" along the other dimension. Results of the chi square analysis are given in table 2.

Table 1
Shorter-Same-Longer Tasks
Means and Standard Deviations of Scores

	Grade 1	Grade 3	Grade 6
8 Slide Tasks (Hoyt Reliability = .82)			
Mean	78.3%	86.5%	91.3%
S.D.	.31	.14	.19
4 Rotation Tasks (Hoyt Reliability = .90)			
Mean	87.5%	87.5%	92.5%
S.D.	.30	.30	.33
1 Flipping Task			
Mean	90.0%	90.0%	88.9%
S.D.	.30	.30	.31
6 One-Figure Tasks (Hoyt Reliability = .87)			
Mean	86.6%	86.7%	93.3%
S.D.	.30	.28	.15
7 Two-Figure Tasks (Hoyt Reliability = .85)			
Mean	78.6%	85.4%	88.9%
S.D.	.32	.14	.26

Table 2
Summary of Chi Square Results for the Shorter-Same-Longer Tasks

Tasks	df	X^2	p less than
Rotate Tasks	1	2.16	.15
Slide Tasks	1	10.995	.001
Flipping Task	1	.299	.70
One-Figure Tasks	1	1.677	.20
Two-Figure Tasks	1	8.191	.005

The results show that the hypothesis may be rejected for two of the sets of tasks. Conservers performed better than non-conservers on the slide tasks ($p < .001$) and on the two-figure tasks ($p < .005$). Both of these task sets are extensions of the standard Piagetian test for conservation of length in that geometric figures (triangles) are used in place of the usual line segments. Thus, the results indicate that students seem to respond consistently on the Shorter-Same-Longer tasks with regard to their performance on the Piagetian length tasks. When a one-way MANOVA was run with student scores on the five different groupings of tasks as the five dependent variables, no significant differences were found between first, third, and sixth graders' performances. Together, the chi square and the MANOVA results seem to indicate that a student's conservation stage (rather than just his grade level) should be taken into consideration when transformation geometry concepts involving invariance of length on slides or involving comparison of two congruent figures are presented.

Before-After Points Tasks

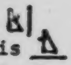
When a triangle is moved by flipping, sliding, or rotating, all of the points on the triangle's image remain in the same relative position to each other as in the original triangle. That is, if point A is on side XY of triangle XYZ, then the image of A will be on the image of side XY; also, the distance between point A and vertex X will equal the distance between the images of point A and vertex X. The series of Before-After Points tasks was designed to determine whether students realized that the three given transformations preserved incidence and distance.

The materials used for this task set included a right triangle drawn on a transparent plastic square, a second right triangle (congruent to the first and in the same position) drawn on a second transparent square, and two pennies. The tasks began with the triangles superimposed. The top triangle then was moved according to a specified transformation. The investigator used a penny to represent one point on the stationary triangle (the "pre-image"). The second penny was given to the student for him to place on the second triangle (the "image") in the spot to which the point would have moved under the given transformation. A total of eight motions were tested: a quarter-turn clockwise, a quarter-turn counterclockwise, a half-turn clockwise, a half-turn counterclockwise, a vertical slide, a horizontal slide, a flip over a vertical axis, and a flip over a horizontal axis.

Ten students in each of grades one, three, and six took the nine-item series of Before-After Points tasks. The KR-20 reliability estimate for this total population was .89. Table 3 gives the percent of students at each grade who answered each item of the Before-After Points tasks correctly.

Table 3

Percent of Students Answering Each Item Correctly

Item	Grade 1	Grade 3	Grade 6
Quarter-turn clockwise	40%	50%	100%
Quarter-turn counterclockwise	60%	60%	100%
Half-turn clockwise	50%	60%	100%
Half-turn counterclockwise	40%	60%	100%
Horizontal slide	80%	80%	90%
Vertical slide	90%	60%	100%
Flip--vertical axis 	80%	70%	100%
Flip--horizontal axis	75%*	60%	100%
Flip--horizontal axis	80%	60%	100%
KR-20 reliability estimate	.70	.92	--

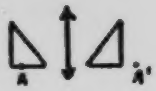
*Only eight students tried this item.

Question 1: What types of errors would students make on these tasks?

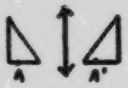
There were a number of errors a student could make. However, since the investigator was interested in seeing whether students seemed to focus on proximity to a vertex and/or on incidence properties in working through the tasks, the following criteria were used for classifying student errors:

The student placed the point's image

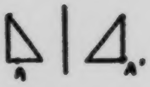
(1) completely off the triangle's image;



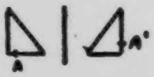
(2) on the correct side of the triangle (excluding the endpoints), but in the wrong location;



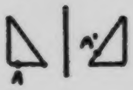
(3) on the image of the closest vertex;



(4) near the correct vertex, but on the wrong side;



(5) in some location other than described in (1) through (4).



In both grades three and six, 10 students took nine Before-After Point tasks, so there was a possibility for a total of 90 errors that could be made. Two of the 10 students in grade 1 did not take one of the horizontal flip tasks, so there was a possibility of 88 errors for the first graders. Results showed that in grade 1, a total of 30 errors were made (34% of the possible number); in grade 3, a total of 34 errors (38%); and in grade 6, only one error (1%). Table 4 shows the classification of the errors made at each grade

Table 4

Classification of Errors on Before-After Points Tasks

	(1) Completely Off	(2) Correct Side	(3) On The Vertex	(4) Near The Vertex	(5) Other
Grade 1	0	18	0	4	8
Grade 3	0	10	10	3	11
Grade 6	0	0	0	0	1

An analysis of theses errors shows a changing pattern of incorrect responses for students at different grade levels. Students in grade 1 most often placed a point's image on the correct side of the triangle, but in the wrong location on that side. Thus, they seem to be focusing on the triangle's sides, rather than on its vertices. Students in grade 3, however, made about the same number of errors with regard to placing a point's image on the correct side but in the wrong location, in comparison to placing the image on or near the correct vertex. Thus, third graders seem to be more aware of both the sides and the vertices of the triangle. And, finally, sixth graders seem to have mastered the task of locating points with respect to both line segment and vertex criteria.

Hypothesis 1: There is no difference between performance of older students compared to performance of younger students on the Before-After Points tasks.

A one-way MANOVA was run on the students' scores in each of the three grades; total scores on rotation tasks (four items), slide tasks (two items), and flip tasks (three items) were used as the three dependent variables being analyzed. The means and standard deviations for each set ot sub-tasks at each grade are given in table 5.

Table 5

Means and Standard Deviations on Rotation, Slide, and Flip Before-After Points Tasks for Students in Grades 1, 3, and 6

	Grade 1	Grade 3	Grade 6
4 Rotation Tasks (Hoyt Reliability = .89)			
Mean	1.90	2.30	4.00
S.D.	1.60	1.83	0.00
2 Slide Tasks (Hoyt Reliability = .44)			
Mean	1.70	1.40	1.90
S.D.	.48	.84	.32
3 Flip Tasks (Hoyt Reliability = .73)			
Mean	2.20	1.90	3.00
S.D.	.79	1.37	0.00

On the results of the MANOVA, the standardized discriminant function showed the rotation tasks score to be the strongest variable (with a coefficient of 1.260), with the flip tasks score next at -.421 and the slide tasks score having a coefficient of .162. The multivariate test of significance using Wilks lambda criterion gave an F of 2.641, which was significant at the .027 level. The univariate F tests gave the following results:

Variable	df_1	df_2	MS	F	p Less Than
Rotation	2	27	12.433	6.334	.006
Slide	2	27	.633	1.819	.181
Flip	2	27	3.233	3.880	.033

Tukey's tests, run to determine the nature of the differences, showed that:

(1) On the rotation tasks, grade one students scored lower than grade six students ($p < .01$), and grade three students scored lower than grade six students ($p < .05$).

(2) On the flip tasks, grade three students scored lower than grade six students ($p < .01$).

Hypothesis 2: There are no differences between performance on the rotation tasks, the slide tasks, and the flip tasks.

Student errors were grouped according to the transformation being tested. Four tasks involved rotations (so a total of 40 errors was possible for each grade), two tasks involved slides (20 possible errors), and three tasks involving flips (30 possible errors at grades three and six, 28 at grade one). Table 6 shows the mean error rate for each grade, classified according to type of transformation.

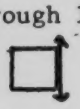
Table 6

Before-After Points Tasks Transformation Mean Error Rate

	Rotation	Slide	Flip
Grade 1	.53	.15	.21
Grade 3	.43	.30	.37
Grade 6	.00	.05	.00

Although it appeared that students performed better on slides and flips than on rotations, a Friedman Two-Way Layout Test run on the data showed that the equal-performance hypothesis could not be rejected. (The same result occurred when the Friedman test was run on the data with grade six omitted.)

Alphabet Tasks

The Alphabet tasks were designed to measure student understanding of the effects of reflections and rotations upon the orientation of a plane figure. Students were shown a letter of the alphabet, printed on a cardboard square, and asked to imagine what that letter would look like after it had been reflected or rotated. The investigator illustrated the specified motion with a blank cardboard square, using the lower right vertex as the center of rotation and turning the square through 180°, or placing a mirror along the right vertical edge of the card  or the top horizontal edge and permitting the student to look at the reflection of the blank card.

For each letter of the alphabet used, a series of four possible images was shown to the student, and the student was to select the correct answer (or, if he thought the correct image was not among the four shown, he drew a picture of the image). Four motions were tested: a reflection over a vertical axis, a reflection over a horizontal axis, a half-turn clockwise, and a half-turn counterclockwise. For each motion, four letters were used: a letter having a vertical axis of symmetry ("A" or "T"), a letter having a horizontal axis of symmetry ("B" or "C"), a letter having rotational symmetry ("N" or "S"), and a letter having no symmetry ("F" or "J"). A total of 40 students (10 students at each of the grades 3, 6, 9, and 11) took the Alphabet tasks. The KR-20 reliability estimate for this total population was .80.

Question 1: What is the effect of a figure's symmetry (or non-symmetry) on the difficulty level when that figure is reflected or rotated?

In the "Item" column of table 7, the set of letters used with each of the transformations is specified. Entries in table 7 show the number of students (out of 10) at each grade who answered each item correctly.

As a further analysis of this data, a two-way ANOVA (grade by letter of the alphabet) was run separately on each of the four motions. For each of the analyses, the unit of analysis was grade (i.e., n=4) rather than student. Tables 8, 9, 10, and 11 show the results. When differences occurred on the "Letter" dimension, Tukey's tests (using the conservative degrees of freedom) were run to determine the nature of the differences. Results showed that:

(1) On Reflect-Horizontal, where letters T, C, J, and S were used, students scored lower on J than on S ($p < .05$) and lower on J than on T ($p < .05$).

Table 7

Alphabet Tasks - Number of Students Answering Each Item Correctly

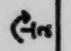
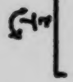
	Item	Grade 3	Grade 6	Grade 9	Grade 11	Total Grades 3, 6, 9, 11
REFLECT-VERTICAL	A	4	9	10	9	32
	B	7	10	8	8	33
	F	6	9	7	10	32
	N	6	10	7	10	33
	C	5	7	10	9	31
	F	2	8	9	10	29
	S	1	0	2	1	4
	T	3	10	9	9	31
REFLECT-HORIZONTAL	T	7	10	9	8	34
	C	2	7	8	6	23
	J	0	6	3	5	14
	S	7	9	7	8	31
	B	6	9	9	9	33
	J	2	9	6	8	25
	N	1	2	2	3	8
	A	6	10	9	9	34
KR-20 Reliability on all 16 items		.51	.53	.40	.79	.80

Table 8

Alphabet Tasks - ANOVA on Reflect-Vertical

Source	df	SS	MS	F	p less than
Grade	3	35.250	11.750	7.437	.01
Letter	3	.250	.083	.053	---
Residual	9	14.250	1.583		
Total	15	49.750			

(Hoyt Reliability of the four Reflect-Vertical items was .59)

Table 9

Alphabet Tasks - ANOVA on Reflect-Horizontal

Source	df	SS	MS	F	p less than
Grade	3	34.250	11.417	6.740	.025
Letter	3	60.250	20.083	11.855*	.005*
Residual	9	15.250	1.694		
Total	15	109.750			

(Hoyt Reliability of the four Reflect-Horizontal items was .35)
*Geisser-Greenhouse conservative F test, df_1=1 and df_2=3, gives $p <$ [illegible]

Table 10

Alphabet Tasks - ANOVA on Half-Turn Clockwise

Source	df	SS	MS	F	p less than
Grade	3	57.688	19.229	6.058	.025
Letter	3	130.688	43.563	13.725*	.005*
Residual	9	28.563	3.174		
Total	15	216.938			

(Hoyt Reliability of the four Half-Turn Clockwise items was [illegible]
*Geisser-Greenhouse conservative F test, df_1=1 and df_2=3, gives $p <$ [illegible]

Table 11

Alphabet Tasks - ANOVA on Half-Turn Counterclockwise

Source	df	SS	MS	F	p less than
Grade	3	35.500	11.833	9.683	.005
Letter	3	108.500	36.167	29.597*	.001*
Residual	9	11.000	1.222		
Total	15	155.000			

(Hoyt Reliability of the four Half-Turn Counterclockwise items was [illegible]
*Geisser-Greenhouse conservative F test, df_1 = 1 and df_2 = 3, gives
$p < .025$.

(2) On Half-Turn Clockwise, where letters T, C, F, and S were used, students scored lower on S than on F ($p < .05$) lower on S than on C ($p < .05$), and lower on S than on T ($p < .05$).

(3) On Half-Turn Counterclockwise, where letters A, B, J, and N were used, students scored lower on N than on J ($p < .05$), lower on N than on B ($p < .01$), and lower on N than on A ($p < .01$).

The results indicate that the action of rotating a figure which already possesses rotational symmetry (the "S" and the "N") was very difficult for students of all age levels to visualize. Students also found the horizontal reflection of the non-symmetric figure "J" to be a difficult task.

Hypothesis 1: There is no difference between performance on tasks involving a reflection over a vertical line of reflection and performance on tasks involving a reflection over a horizontal line of reflection.

Each student's total score on the four reflect-vertical items and on the four reflect-horizontal items was used as the dependent variables. A one-between, one-within ANOVA showed a grade level difference in performance on all of the tasks, but also showed no significant difference between student performance when the line of reflection was vertical and when it was horizontal.

Hypothesis 2: There is no difference between performance on clockwise half-turn items and performance on counterclockwise half-turn items.

Each student's total score on the four clockwise half-turn items and on the four counterclockwise half-turn items was used as the dependent variable. A one-between, one-within ANOVA again showed a difference in performance related to grade level. However, results also showed that the direction of the rotation did not seem to affect student performance.

Hypothesis 3: There is no difference between performance of older students compared to performance of younger students on the Alphabet tasks.

A one-way MANOVA was run on student's scores in each of the four grades; total scores on Reflect-Vertical, Reflect-Horizontal, Half-Turn Clockwise, and Half-Turn Counterclockwise were used as the four dependent variables being analyzed. The multivariate test of significance using Wilks lambda criterion gave an F of 3.981, which was significant at the .001 level. The univariate F tests gave the following results:

Variable	df_1	df_2	MS	F	p Less Than
Reflect-Vertical	3	36	4.700	5.755	.003
Reflect-Horizontal	3	36	4.567	5.830	.002
Half-Turn Clockwise	3	36	7.692	12.417	.001
Half-Turn Counter-clockwise	3	36	4.733	4.760	.007

Tukey's tests were run to determine the nature of the differences, with the following results:

(1) Reflect-Vertical--grade 3 students scored lower than students in grade 6 ($p < .05$) and in grade 11 ($p < .05$).

(2) Reflect-Horizontal--grade 3 students scored lower than students in grade 6 ($p < .01$), in grade 9 ($p < .05$), and in grade 11 ($p < .05$).

(3) Half-Turn Clockwise--grade 3 students scored lower than students in grade 6 ($p < .01$), grade 9 ($p < .01$), and grade 11 ($p < .01$).

(4) Half-Turn Counterclockwise--grade 3 students scored lower than students in grade 6 ($p < .01$), grade 9 ($p < .05$), and grade 11 ($p < .05$).

Discussion and Conclusions

The purpose of this study was to provide some base line data on student understanding of some of the basic transformation geometry concepts and to search for patterns that might characterize the development of these understandings. The intent was to be descriptive so that the beginnings of a profile of student understanding could be made, which, when supplemented by further research, might be useful to people engaged in curriculum development efforts.

Results on the Shorter-Same-Longer tasks showed that Piagetian stage (rather than the age of the students) seemed to make an important difference when elementary students considered the question of invariance of length of geometric figures under given transformations. For rotations and flips, most students seemed to believe that length remained invariant, but for slides, the non-conservers saw the length of a geometric figure as changing. Similarly, for tasks involving a before-after comparison of only one figure, most students said that length would stay the same; however, when there was a congruent copy close by with which the student could make a visual comparison, non-conservers were significantly more apt to believe that a transformation changed the length of a figure. When student performance was analyzed according to grade level, no differences among performances were found. These results seem to indicate that curriculum developers might consider tailoring transformation geometry content for elementary students according to Piagetian stage, rather than grade level.

Results on the Before-After Points tasks showed a pattern to student errors. First graders most often located a point's image on the correct side of a triangle but in the wrong location on that side. Third graders made about as many errors as the first graders did on these tasks, but the nature of their errors differed from the first graders. While about one-third of the errors came when the third graders placed the image of a given point on the correct side of the triangle but in the wrong location on that side, another one-third of the errors came from placing the image on or near the correct vertex. The third graders seemed to be much more aware of both the sides and the vertices of a triangle than the first graders, who concentrated mainly on the triangles' sides. And sixth graders, with only one error on all of the tasks, seemed to have mastered the problem of using both the vertices and the sides of a triangle as cues in locating the image of a point. Altogether, these results appear to support the conjecture that the first features to be discriminated may be roughly classed as topological (such as edgedness, proximity, and configuration) as opposed to Euclidean (such as corners) (Olson, 1970; Piaget & Inhelder, 1967; Martin, 1976).

Results on the Alphabet tasks fit in with Schultz's work (1976)[*], showing that the difficulty level of a transformation task is affected by the attributes of the configuration used and that the difficulty will vary from isometry to isometry and from age group to age group. In the case of the Alphabet tasks, half-turns on figures possessing rotational symmetry were consistently very difficult for students in all of the grade levels tested. (By far, the most popular incorrect responses were the image "Ƨ" for the half-turn on "S" and the image "И" for the half-turn on "N.") The horizontal reflection of the non-symmetric figure "J" also was a problem for many of the students. A difference was found in overall performance on the Alphabet tasks related to age, with third graders scoring significantly lower on all of the four motions tested than students in grades 6, 9, and 11. The analyses, however, did not show any differences between student performances on reflections over a vertical line of reflection compared to a horizontal line of reflection, nor any differences between performances on clockwise half-turns compared to counterclockwise half-turns.

Sequencing and structuring transformation geometry topics with regard to what is known about student understandings are long-range goals. The findings reported here (which, of course, need to be confirmed or modified by further research using larger populations of students and using students at intervening grade levels) suggest beginnings of patterns of student understandings which need to be explored and expanded.

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[*] For a summary, see the article by Schultz in this monograph.

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Variables Influencing the Difficulty of Rigid Transformations During the Transition Between the Concrete and Formal Operational Stages of Cognitive Development

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This paper is about the relationship between the mathematical and cognitive structures as evidenced by performance on rigid transformation tasks by children who are in the transition between the concrete and formal operational stages (in the Piagetian sense) of cognitive development. Several factors contributed to the conception of this investigation. Piaget's interest in logico-mathematical thinking in children has been expressed predominantly throughout his work with children who are at the end-points of the concrete operational stages (Flavell, 1963), thus avoiding the age groups that are in between. Furthermore, Piaget has not dealt with mathematical topics that are usually taught in elementary school. Therefore, this study was designed to investigate the transitional phases that children go through between the beginning of the concrete operational period and the beginning of the formal operational period; and it was designed to investigate a concept area that _is_ being incorporated into the elementary school curriculum--transformation geometry.

Little is yet known about or agreed upon regarding children's cognitive abilities concerning transformation geometry. Consequently, isolation of such abilities is an important area of investigation from the point of view of theory building as well as instructional application. Piaget reminds us that cognitive growth is a gradual mastery of invariance properties under systems of transformations (Piaget & Inhelder, 1971). Thus, it may be possible to find techniques to anticipate the relative difficulty of transformation geometry concepts. Similar techniques could then be used to organize sequential presentation of arithmetic ideas--or instructional models leading to arithmetic ideas. Lesh (1976) makes the following statement about an existing technique.

> "One of the cornerstones of Piagetian theory rests on the psychological viability of analyzing and equating tasks (and concepts) on the basis of their underlying operational structures. Yet, transformation tasks which are operationally isomorphic often vary widely...in degree of difficulty. Consequently, if operationally isomorphic tasks differ 'too much,' it may be meaningless to equate tasks on the basis of operational structure." (p. 228)

One of the major purposes of this study was to isolate criteria for anticipating the relative difficulty of various transformation geometry tasks. It was hypothesized that the complexity of configuration and the complexity of transformation displacement, in addition to operational structures, affects the difficulty of transformation tasks for children.

Piaget (Laurendeau & Pinard, 1970) has described several factors that account for décalages involving operationally isomorphic tasks that differ only in figurative content. However, Piaget has generally tended to deemphasize the importance of décalages, preferring to focus on the analysis of ideas rather than on the analysis of concrete materials and figurative models that embody those ideas.

In the introduction to Laurendeau and Pinard's "Space" book, (Laurendeau & Pinard, 1970), Piaget stated:

> "Décalages derive from the object's resistances, and the authors (Laurendeau and Pinard) ask that we construct a theory of these resistances, as though this were an undertaking directly parallel to the one concerning the subject's actions and operations. It is an exciting project and it should certainly be considered. But we must remark at once that if the subject's actions always reflect intelligence (a condition which greatly facilitates the analysis), the object's resistances do not do so, and involve a much greater number of factors." (p. 4)

However, even if psychologists believe they can ignore variations due to figurative content, the issue is highly important to educators who must use concrete materials and figurative models to teach mathematical concepts. The present study was an attempt to identify the resistances offered by the complexity of the transformation displacement as well as those offered by the complexity of the configuration transformed.

Review of the Literature

Piaget and Inhelder (1971) did several studies relating to this investigation. The first focused on the operative aspect of transformation tasks. It was found that the ability to anticipate the result of a transformation varied substantially among rotations, circumductions, and overturnings, which were three types of rotations. Another study, though not on transformations but rather representations, looked at the effect of the subject's involvement in the construction of the configuration. It was found that a configuration a child constructed was more easily and more correctly reproduced than one with whose construction she/he had less involvement. This suggests that a certain familiarity with the configuration was afforded through the construction of it. In another study Piaget and Inhelder also concluded that complexity of the operative aspect of a transformation task had more of an effect on difficulty than newness of a configuration did. Finally, in a study regarding length of translation displacement, subjects were unable to anticipate the image of an overlapping translation; or, if they were able, they repeatedly distorted the images represented.

Although a "time lag" in the ability to perform operationally equivalent tasks is evident in these studies, the results are usually interpreted as providing supporting evidence that the operative aspect of a transformation task is dominant over the figurative aspect. The one exception, the study covering involvement in constructing the configuration, demonstrates the important influence of familiarity of configuration. This is significant and suggests a possibility of other causes of familiarity besides assistance in construction.

Thomas (in present monograph) investigated the relative difficulties within one transformation, between two, and among more than two transformations. For example, she studied the differences in difficulty between clockwise and counterclockwise rotations. Thomas also compared the effect on difficulty of reflecting a plane figure versus a point. This study dealt primarily, though, with the variations of the operative aspect of a transformation. The comparison between plane figures and points is essentially a comparison of dimensionality and could be more telling if an additional comparison were made with three-dimensional figures. At any rate, this still suggests other common influencing factors which haven't been considered, such as size or meaningfulness.

Moyer (1974) found that explicit knowledge of the physical motion associated with a tranformation didn't necessarily help the child's ability to perform a transformation task. Yet, Perham (in present monograph) found that horizontal displacements in translation and rotation tasks were significantly easier for first grades than the same tasks involving diagonal displacements. Perham's and Thomas' studies are interesting in the light of Moyer's findings. If knowledge of motion is not necessary, why did differences in translation direction affect performance? And why should it matter to look at clockwise and counterclockwise rotations? If the end points are the same, should it matter which direction a rotation takes? Motion to an adult mind implies directionality and dimensionality. Does it to a child's mind?

Kidder (1976) dealt more or less directly with the issue at hand. He observed 9, 11, and 13 year old subjects who had not yet reached the formal operational stage demonstrate the inability to sort out influencing factors while performing transformation tasks. He looked at individual motion, composition, and inverse motions at the representational level and found that the ability to perform transformations at this level seems to develop from formal operational thought.

The question remained as to whether task attributes could be modified enough so even younger subjects would be successful with performance. A closer look could be made at the logical formalization of assimilating and accommodating the complexity factors of displacement and configuration. Kidder did not concentrate on specific features of either.

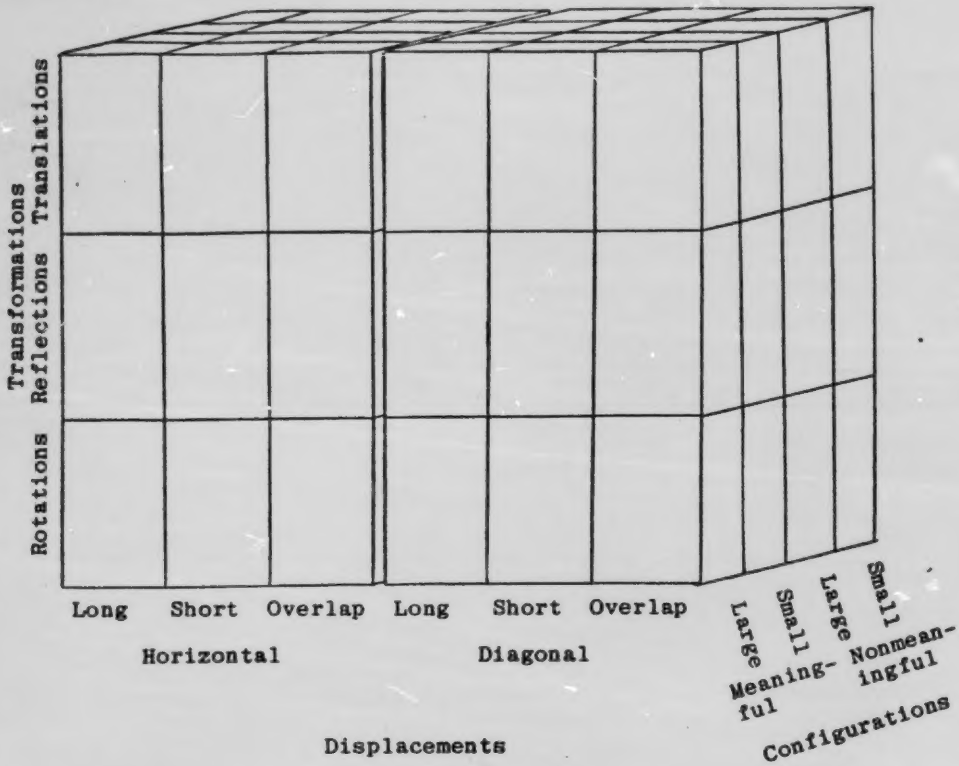
In summary, these studies focused on (a) the operative attributes of transformation tasks rather than on both the operative and the figurative attributes. Further, (b) variations of an attribute were generally not considered, and (c) a given attribute was generally not considered under each of the three isometries. The later trend was probably due to assuming that operational isomorphism is a basis of considering all translations similar to each other in difficulty, and so on for reflections and rotations.

For the present study, three pilot studies (See Schultz, 1976) determined which attributes of the displacement and the configuration were most influential. Variations of these attributes were determined and then built into each isometry to determine effect on task performance.

Method

Complexity of transformation tasks was varied in direction and size of displacement, and in familiarity and size of configurations. Figure 1 represents the over-all design for determing tasks.

Figure 1. Design for determining tasks.


Transformations
Translations
Reflections
Rotations
Long
Short
Overlap
Long
Short
Overlap
Horizontal
Diagonal
Displacements
Large
Small
Large
Small
Meaning-ful
Nonmean-ingful
Configurations

The directions of displacement were horizontal and diagonal* as illustrated in Figure 2.

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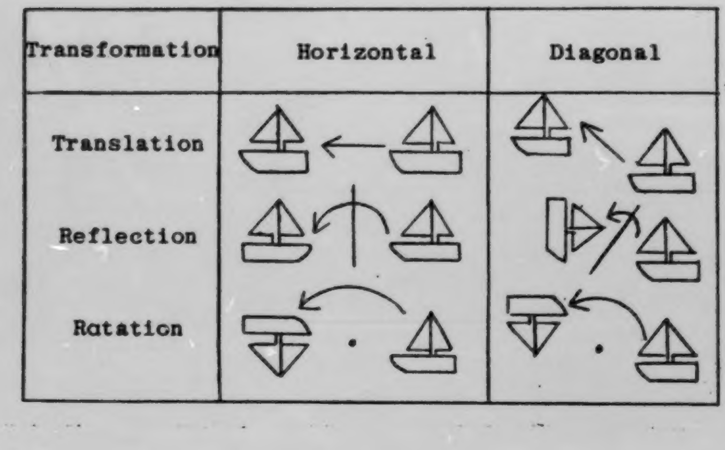
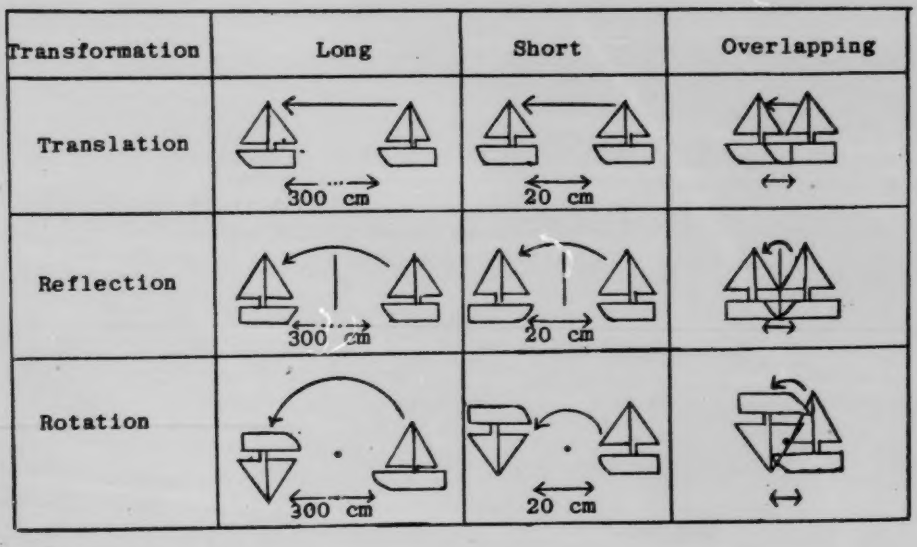
* Horizontal ← (or →) versus vertical ↑ and diagonal ↗ versus diagonal ↖ comparisons were made but are not presented in this paper. For the comparison results, see Schultz (1976).

Figure 2. Directions of configuration displacement.


Transformation
Horizontal
Diagonal
Translation
Reflection
Rotation

This distinction was determined by the tendency to organize the real world on a horizontal-vertical coordinate system (Hubel, 1972). Size of transformation displacement referred to the distance between the initial and final locations of the configuration. The long distance was 300 cm, the short distance was 20 cm, and the last was an overlap of half the configuration, as illustrated in Figure 3.

Figure 3. Size of transformation displacement.


Transformation
Long
Short
Overlapping
Translation
300 cm
20 cm
Reflection
300 cm
20 cm
Rotation
300 cm
20 cm

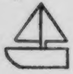
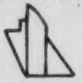
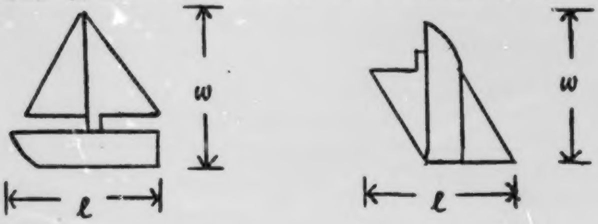
Familiarity of configuration referred to the meaningfulness of a picture designed by the way three shapes* were arranged. One arrangement produced a meaningful configuration (, illustrated in Figures 2 and 3) and the other nonmeaningful (). In general, a meaningful configuration represented something real and commonplace, whereas a nonmeaningful configuration did not. This distinction was formulated on the basis of pilot study results and confirmed in the present study's pre-interview questions. Each subject was asked what looked like. Responses were always "sailboat, boat, or ship," whereas there was no response when shown the other configuration. Size of configuration referred to the measure in centimeters of the length and width of the configuration. The large configuration was 80 cm by 80 cm, and the small configuration was 8 cm by 8 cm. Figure 4 shows that length equaled width for each configuration.

Figure 4. Length was equal to width in both configurations.



Population. There were 40 six year olds, 80 seven year olds, 70 eight year olds, 70 nine year olds, and 10 ten year olds from Martin Luther King, Jr. Laboratory School in Evanston, Illinois. This school maintains the same socio-economic and racial proportions as Evanston by selective acceptance of children. In this respect the subjects of this study were unique, since they had to have parents with the initiative to apply for their children's admittance. Nevertheless, subjects were randomly chosen from each age group regardless of learning disabilities, intelligence quotients, achievement scores, or grade in school.

Subject's role. Subjects were interviewed individually by the experimenter. Each interview began with a brief verbal exchange and pre-task activities to assure the experimenter that the subject understood the basic rules of the "game" which was about to be played. Then each subject did a

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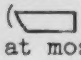
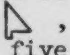
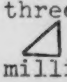
* Each configuration consisted of three contiguous, though nonoverlapping, distinct wooden shapes (, , ) having length, width, and thickness -- the last being at most five millimeters. The shapes were also distinct in color: was red, yellow, and blue.

task set consisting of a sampling of from 8 to 10 tasks.

Since this study was concerned with the representational level of operational comprehension in the Piagetian sense (Piaget & Inhelder, 1967), the tasks were designed to elicit a mental operation on a mental image of a concrete configuration. That is, subjects had to imagine the configuration's transformed position, and then to represent this correctly with a physical model. (This procedure, however, was different from Kidder's as reported in this monograph, since in this case tasks were modified according to attribute variations anticipating successful performance by younger subjects). To do this, the subject first observed the transformation of a large square sheet of plexiglass which originated from the top of the experimenter's configuration (mounted on another sheet of plexiglass so that the arrangement was fixed for convenience). Then the subject took his/her configuration shapes and placed them in the exact position on the transformed plexiglass to show how the experimenter's configuration would look if it moved exactly the same way as the plexiglass. A subject could vary the shape of a configuration, but not the size. That is, "distractor" pieces were not made available to the subject as they were in Kidder's investigation, which is recorded in this monograph.

Six year olds were each given a sampling of tasks from the pool of translation tasks only. A major assumption made at this point was that if a task were do-able by a particular age group, it would also be do-able by the next older age group. Therefore, seven year olds were each given a sampling from the pools of reflection tasks, rotation tasks, and translation tasks that the six year olds were unable to do; and so on. The ten year olds were given only nine hand-picked tasks, only one of which is reported.

Each task set was done by 10 different subjects, so each of the 72 cells (See Figure 1) collected 10 responses. Each set of about 10 tasks that a subject did was randomly determined.

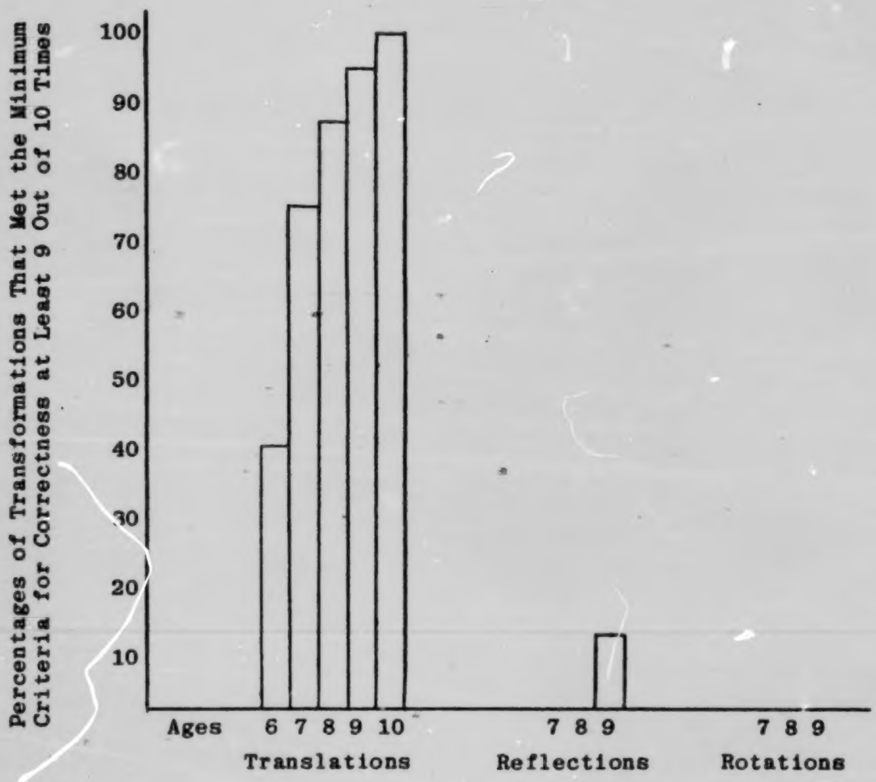
Collection of data and analysis. The subject's exact arrangement was recorded for each transformation task in the set and the tasks were evaluated for correctness by the experimenter immediately upon completion of the entire task set.

A task that was determined do-able for a particular age group had to be performed at least 9 out of 10 times correctly by children in that age group. The criteria for correctness were: invariance of configuration, spatial orientation of entire configuration, and the order of the two triangular shapes. Each response was judged on each criterion independently of the other two criteria. An elaborate procedure was designed for judging each response on each of these criteria. Also, since a response was not just judged right or wrong, a further judging procedure was employed for identifying minimum and absolute correctness of each response. For complete descriptions of these procedures and for additional data not shown here, see Schultz (1976).

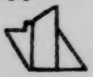
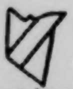
Results and Discussion

Two kinds of data are described: (1) correctness of task responses, and (2) do-ability of tasks. Recall that correctness can be minimum or absolute; and do-ability of a task means that it was minimally correct no fewer than 9 out of 10 times it was performed. Figure 5, which reports do-ability of all tasks in each isometry classification, shows that translation tasks were exceedingly more do-able than the reflection and rotation tasks. In fact, there is no comparison between the difficulty of translations and the other two isometries, whereas the other isometries are comparable. Hence, correctness and do-ability data are presented for (a) translations (Table 1 and Figure 6) followed by discussions of the most significant error patterns that emerged. The same is then presented for (b) reflections and rotations together (Table 2 and Figure 7). Each discussion is determined by the attributes of the transformation tasks.

Figure 5. Comparison of do-ability of transformations across all attributes.



Attributes of Translation Tasks

Direction of Translation. There is a noticeable indirect proportion between do-ability (Figure 6) of diagonal translations and minimum correctness (Table 1) of these tasks. Perhaps what occurred is an even distribution of difficulty across attributes for horizontal tasks for the six year olds, so that more tasks were nondo-able. This can be observed more closely by the raw data (See Schultz 1976). Six year old correctness reresults correlate with Perham's findings (in present monograph) for first graders. She found that diagonal translations were significantly more difficult for first graders to master than horizontal translations even after instruction. In fact, in the present study it was found that the kinds of errors made by six year olds with diagonal displacements in translation tasks were different in nature than those involving horizontal displacements. Subjects centered on the direction of the displacement to the extent that they ignored an invariance propoerty of the translation, which is spatial orientation of configuration. It was often the case that the configuration appeared to be headed in the direction of the displacement. For example, if  were translated ↗, the response might be .

Size of Translation. It is quite clear that the short translations were far easier than the long or overlapping translations. The spatial orientation errors made with long translations were centering on the direction of the displacement, regardless of whether it was in a diagonal or a horizontal direction. The configuration image was rotated to head in the direction of the movement. There was apparent inability to view both the experimenter's configuration and the representational mental image in one glance--thus the misrepresenting of the configuration.

Errors due to overlapping displacements were mostly accountable to the confounding of the image by the experimenter's configuration. Very often the subject took the liberty to push the overlapping transformed plexiglass off the experimenter's configuration before representing the image. In the cases where the plexiglass was not pushed off the experimenter's configuration, the image was frequently positioned partially off the sheet of plexiglass, horizontally away from the experimenter's configuration. This way there was no overlapping of the configurations. There was definitely a struggle to represent the image when it overlapped with the experimenter's configuration, which agrees with Piaget and Inhelder's study on overlapping translations.

Familiarity of Translation Configuration. As occurred with horizontal and diagonal translation results for six year olds, there is an indirect proportion between the six year olds' performance with meaningful and nonmeaningful configurations. Again, perhaps the difficulty across attributes for meaningful configuration tasks was uniformly difficult for six year olds, causing more tasks to be "slightly" nondo-able, instead of fewer tasks being "very" nondo-able. Common was constructing a meaningful

Table 1

Percentages of Translation Tasks Whose Responses Met the Absolute Criteria for Correctness and the Minimum Criteria for Correctness

Attribute	# of tasks	% Abs.Crit. % Min.Crit.	# of tasks	% Abs.Crit. % Min.Crit.		# of tasks	% Abs.Crit. % Min.Crit.		# of tasks	% Abs.Crit. % Min.Crit.		# of tasks	% Abs.Crit. % Min.Crit.	
Horizontal	12	61.7 83.3	8	52.5 76.2	72.5 87.5	2	45.0 60.0	75.0 85.0	1	60.0 80.0	80.0 100	0		
Diagonal	12	55.8 79.2	6	50.0 68.3	70.0 78.3	4	60.0 67.5	67.5 72.5	2	40.0 50.0	65.0 75.0	1	40.0 60.0	90.0 90.0
Long	8	52.5 73.8	7	52.8 71.4	60.0 75.7	4	47.5 60.0	57.5 67.5	3	46.7 60.0	70.0 83.3	1	40.0 60.0	90.0 90.0
Short	8	65.0 90.0	2	40.0 75.0	90.0 100	0			0			0		
Overlap	8	58.8 80.0	5	54.0 74.0	80.0 88.0	2	70.0 75.0	95.0 95.0	0			0		
Meaningful	12	66.7 80.8	8	58.8 73.8	77.5 85.0	4	65.0 72.5	85.0 85.0	1	60.0 60.0	90.0 90.0	0		
Nonmeaningful	12	50.8 81.7	6	41.7 71.7	63.3 81.7	2	35.0 50.0	40.0 60.0	2	40.0 60.0	60.0 80.0	1	40.0 60.0	65.0 90.0
Large	12	63.3 84.2	6	55.0 76.7	81.7 95.0	1	70.0 70.0	60.0 60.0	1	60.0 60.0	90.0 90.0	0		
Small	12	54.2 78.3	8	48.8 70.0	63.8 75.0	5	52.0 64.0	72.0 80.0	2	40.0 60.0	60.0 80.0	1	40.0 60.0	90.0 90.0
Ages		6		6	7		7	8		8	9		9	10

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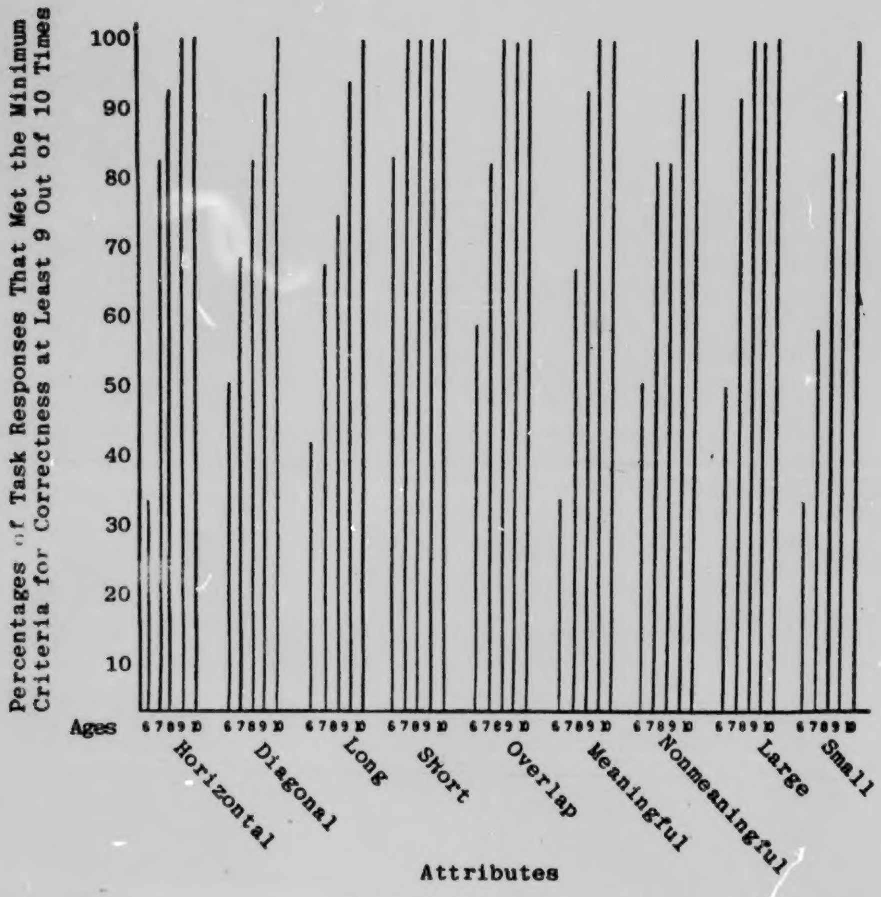
Figure 6. Do-ability of translation tasks for six through ten year olds.

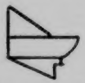
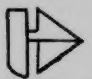
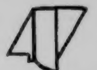
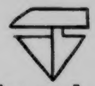
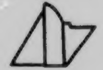
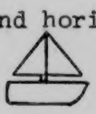


configuration for a nonmeaningful one. The subjects seemed to more spontaneously recall the meaningful picture. Meaningfulness of a configuration apparently facilitated the operational comprehension of a task. There was an obvious enthusiasm among subjects toward the large, meaningful configuration. The meaningful and nonmeaningful configurations elicited comparable spatial orientation errors in translation tasks, except by the eight year olds. They made more spatial orientation errors with the nonmeaningful configurations than with the meaningful.

Size of Translation Configuration. Large configurations were far easier to translate than small configurations for every age group except age eight. It is not immediately obvious to the experimenter why more errors occurred with small configurations. Perhaps an investigation of imitation construction would be appropriate. It is not unreasonable to suspect that an operational disequilibrium occurs with configurations too small. It might be important, though, to find out what is "too" small. The errors that did emerge were possibly due to the fact that the smaller the configuration, the greater the distance between the edges of the configuration and the walls of the testing room. This possibly caused "losing touch" with points of reference for determining spatial orientation.

Attributes of Reflection and Rotation Tasks

Percentages of tasks that were minimally and absolutely correct are again listed. Each percentage is based on the same number of tasks across ages and isometries, since each age group did all the tasks. Since no rotation tasks were do-able at any age, a do-ability graph for rotations is not shown.

Direction of Reflection and Rotation Displacement. The percentages represented in Table 2 show the most dramatic effect of an attribute on the difficulty of any isometry. The most significant error pattern that emerged with diagonal reflections concerned spatial orientation. The diagonal reflection tasks, for which the correct response was  or , were sometimes responded to with  or , which indicates a fixation on the vertical displacement exclusively and not the combination of the vertical ↑ and horizontal ← resulting in a diagonal ↖. Also,  and  were the responses for the same tasks, representing a fixation on the horizontal ← displacement exculsively. However, it could be that the fixation was just on the "flipness" of this isometry. Similar errors with rotations occurred, but there were slightly fewer spatial orientation errors with diagonal displacements than with horizontal ones.

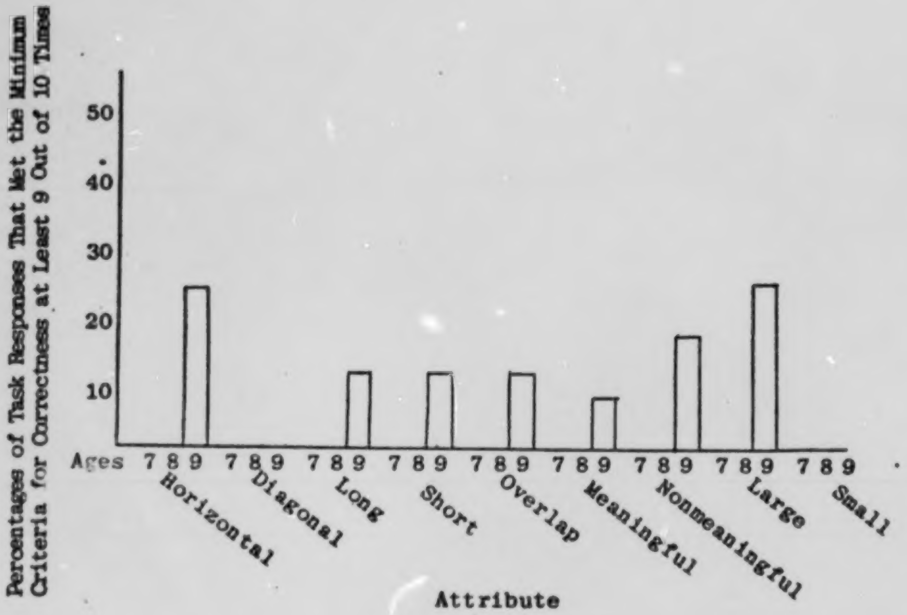
Size of Reflection and Rotation Displacement. A glance at the absolute and minimum criteria percentages for both reflections and rotations involving this attribute (Table 2) gives the impression of very similar results. Figure 7 shows that do-ability of reflections does exceed all the do-ability of rotation tasks within these attributes. (Recall that no rotation tasks were do-able). There were no construction errors by seven year olds with short reflections. The reason for this seems obvious--the subject had no

Table 2

Percentages of Reflection and Rotation Tasks Whose Responses Met the Absolute Criteria and the Minimum Criteria for Correctness

Attribute	# of tasks	Reflections			Rotations		
		% A.Crit / % M.Crit	% A.Crit / % M.Crit	% A.Crit / % M.Crit	% A.Crit / % M.Crit	% A.Crit / % M.Crit	% A.Crit / % M.Crit
Horizontal	12	25.8	45.8	59.2	16.7	22.5	29.2
		37.5	60.8	70.8	27.5	36.7	42.5
Diagonal	12	4.2	4.2	5.8	12.5	11.7	35.0
		6.7	6.7	10.8	19.2	16.7	49.2
Long	8	13.8	23.8	26.2	15.0	21.2	28.8
		21.2	36.2	41.2	21.2	30.0	36.2
Short	8	16.2	28.8	37.5	15.0	17.5	32.5
		23.8	36.2	41.2	23.8	27.5	46.2
Overlap	8	15.0	22.5	33.8	13.8	12.5	35.0
		21.2	28.8	40.0	25.0	22.5	55.0
Meaningful	12	16.7	31.7	32.5	17.5	21.7	46.7
		20.8	35.0	37.5	24.2	30.8	54.2
Nonmeaningful	12	13.3	18.3	32.5	11.7	12.5	17.5
		23.3	32.5	44.2	22.5	22.5	37.5
Large	12	16.7	22.5	33.3	15.0	10.0	34.2
		25.0	31.7	45.0	25.0	16.6	45.0
Small	12	13.3	27.5	31.7	14.2	24.2	30.0
		19.2	35.8	36.7	21.7	36.7	46.7
Ages		7	8	9	7	8	9

Figure 7. Do-ability of reflection tasks for six through ten year old subjects.



difficulty viewing in a single glance the experimenter's configuration and a mental representation of the image in its new location.

The eight year olds found the overlapping reflections most difficult. The problem seemed to be mostly with identifying correct spatial orientation. In general, it could be said that the overlapping reflections tended to be harder than the long and short reflections across the ages, though not by much for the seven and nine year olds.

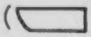
Rotations did not elicit any significant error pattern. In fact, for some reason the eight year olds were markedly successful with ordering the triangular shapes for long rotations.

Familiarity of Reflection and Rotation Configuration. Figure 7 shows that 17% of the reflections involving nonmeaningful configurations were do-able by the nine year olds compared to only 6% for the reflections involving meaningful configurations for the nine year olds. Seven and eight year olds were not able to "do" the reflection tasks at all. Rotation task responses were uniformly more correct across ages when meaningful configurations were involved. As one subject put it when commenting on a meaningful configuration, "It's a hard design; that's what made it hard (to do)."

The most striking error pattern with reflections and rotations concerned spatial orientation. The subjects were certainly able to comprehend the "turnness" of the rotation, but they tended to turn the configuration image so that it faced the direction of the turn. Errors of this kind made with rotation tasks involving nonmeaningful configurations were always slightly more frequent than with meaningful configurations.

Size of Reflection and Rotation Configuration. Table 2 demonstrates that seven and nine year olds found reflecting large configurations easier. The nine year olds' responses to rotation tasks were minimally correct fewer times with large configurations than with small configurations. Ironically, however, these nine year olds responded with absolute correctness using the large configurations more often than with the small.

The difficulty incurred with small configurations in reflection and rotation tasks seemed to be of the same nature as that with translation tasks. The size of the small configurations wasn't even as small as most configurations that appear in elementary mathematics textbooks, on which children are expect to operate.

Behavioral Observations. (1) The design and position of the red () piece had an influence on the approach taken to a response. It was usually positioned first. This is to say that there are influencing internal attributes of meaningfulness and size. (2) Younger subjects did not readily take interest in turning over individual shapes even when they felt a need for a "better fit" in their construction. This behavior might be a function of experience with puzzle pieces which are generally "good" only on one side. (3) There was a frequent disregard for the motion involved in a task. One subject explained, "It doesn't matter how you moved it, but how you landed it." (4) There was a considerable lack of spontaneous verbalization while performing tasks. In general, this might be said to typify behavior when there is no language available to verbalize an activity.

Conclusions

The term "difficult" has been used throughout this study, and it appears that an operational definition of it has emerged. Certain internal (configuration construction) and external (spatial orientation) structures of a response that were preserved under a transformation were not always preserved under the same transformation involving an attribute variation. So it can be said that a transformation performed in which certain topological or projective-Euclidean properties are preserved becomes difficult when those properties are no longer preserved due to a variation in the figurative or operative aspect of the transformation involved.

Second, task attribute variables of displacement directions and sizes as well as configuration familiarity and sizes were indeed separate causal contributions to disequilibrium throughout the growth of operational comprehension. This is to say that operational structure is not the only determining factor in predicting difficulty.

Third, this study showed that attributes of fixed states (familiarity and size) can be influential enough to interfere with comprehension. However, the features of the operation itself tended to be most influential and contributed the greatest threat of interference.

Generally speaking, these results apply across all curriculum development. A structure should not be imposed upon an activity without first considering which features of action and materials can cause disequilibrium. When a teacher sequences the presentation of two concepts, the child's variability in performance due to influencing attributes must be taken into consideration. Imposed structures, i.e., content or pedagogical, might not necessarily be concurrent with cognitive structures.

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Conservation of Length: A Function of the Mental Operation Involved

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In analyzing measuring capabilities of children, Piaget (Piaget, Inhelder, & Szeminska, 1960) posits: "Underlying all measurement is the notion that an object remains constant in size throughout any change in position" (p. 90). That is, the length of an object which is moved (either physically or mentally) must remain invariant. Piaget calls this cognitive phenomenon "length conservation." Piaget's experimental technique in examining the child's conservation of length capabilities (hereinafter referred to as the "classical" definition of length conservation) consisted of showing the child two little sticks of the same length (approximately 5 cm long) arranged side by side, a few millimeters apart, with their endpoints in exact alignment; asking him if they were equal in length; sliding one of the little sticks forward approximately 1 cm; and again asking the child to say which of the two sticks were longer, or if they were still the same length (Piaget, Inhelder, & Szeminska, 1960). According to Piagetian literature, conservation develops in three stages: an initial stage in which perceptual factors determine the judgement of length, an intermediate stage where both perceptual and conceptual considerations influence the child's judgement, and the stage of consistent conservation. To Piaget, the stages reflect the transition from perceptual intuition to operationality which characterizes the child's cognitive development. In his experiments, Piaget does not give the ages of all the children that he found in each stage; however, the protocols indicate that six to eight years of age was the range of consistent conservation.

Piaget's results concerning stages of development and age range for consistent conservation of an attribute have been verified through replications. Other investigators have gone beyond mere replications which used identical or similar tasks and have examined factors which might effect length conservation and related concepts. In investigating interrelationships among various properties of length relations, Steffe and Carey (1972) define conservation of length in terms of the relations "longer than," "shorter than," and "the same length as." After assuming that a curve can be straightened, they state: "A length relation between two curves A and B is conserved by a child if, and only if, the relation is (a) established by the child, and (b) retained, regardless of any length preserving transformation on one or both of the curves" (p. 81). In examining logical operations and the concept of conservation in children, Shantz and Siegel (1967) suggest that conservation may be viewed as the child's increasing ability to differentiate between reality and appearance, or between relevant and irrelevant attributes. Together these strongly suggest that the common usage of the word "conservation" may be suspect. That is, that the statement, "once a child conserves length (in the classical sense), he always conserves length," may be open to question. Steffe and Carey's definition admits length preserving transformations which differ from the simple slide of the classical test, and Shantz and Siegel introduce the possibility of a child's failing to conserve length because he concentrated on another attribute or factor of the task. In discussing the "décalages" between conservation of mass, weight, and volume, Lesh (1975) notes that "...many psychologists have made the error of attempting to describe the acquisition of "conservation"--as though conservation of what (i.e., mass, weight, or volume) were unimportant" (p. 67). It appears that a similar type error is being made in assuming that once a child conserves an attribute (as evidenced by Piagetian tasks), he will continue to conserve this attribute in other situations. Hence, the present study was devised to examine the experimental hypothesis: "Children who conserve length in the classical sense may lose (or ignore) this length conservation capability while performing a more complex task."

Review of Literature

Conservation experiments (number, area, mass, etc.) abound in the literature. In the monograph arising from the "Measurement" workshop sponsored by the Georgia Center for the Study of Learning and Teaching Mathematics, Carpenter (1976) summarizes experiments relating to conservation and notes: "Piaget's description of the development of conservation has generally been confirmed using a great variety of experimental procedures, materials, and types of transformations" (p. 52). Even though relatively few of the experiments were specifically directed to conservation of length per se, some do imply that young children, ages six to eight, conserve length in the classical sense. Sawada and Nelson (1967) found the threshold for the emergence of length conservation to be between five and six years of age. Diver's (1970) and Steffe and Carey's (1972) studies infer that since children attain transitivity of length at about age seven, they are therefore able to conserve length by this age. Shantz and Smock's (1966) findings concerning the child's development of distance conservation and spatial coordinate system also infer that young children conserve length in the classical sense.

In striking contrast to the results of replications and to related studies which infer that children, ages six to eight, do conserve length, Kidder (1976) found that the majority of 9, 11, and 13 year old adolescents do not conserve length while performing Euclidean transformations at the representational level.* After operationally defining the three basic Euclidean transformations, Kidder used Piagetian-like tasks to test his hypothesis that adolescents could perform Euclidean transformations, inverse transformations, and compositions of transformations at the representational level. Exemplifying the testing procedure is the individual motion task wherein each subject was given an original figure (a triangle), an indicated motion, and seven little sticks (three for construction of a congruent image and four of differing lengths as distractors). Each subject was then asked to construct the image of the original figure as it would look to him after performing the indicated motion. A prototypical task is given in figure 1.

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* Throughout this paper, the word "representational" is used in a strict Piagetian context. To Piaget, the child's spatial development proceeds at two distinct levels, the perceptual and the representational. "Perception" in this sense is the knowledge of objects resulting from direct contact with them, whereas "representational" space requires imagination or thought (Piaget and Inhelder, 1967). Two types of activities exemplify performing Euclidean transformations at the perceptual level: (1) the child physically performs the transformation with cutout patterns or tracings of the original figure; and (2) the child is given a static figure F, an indicated motion, and the static image figure F' congruent to F. In sharp contrast to the perceptual activities, comprehension of transformations at the representational level consists of forming a mental image of the original figure, performing a mental operation (Euclidean transformation) on this representation, and then being able to imagine the figure at rest in its final fixed position.

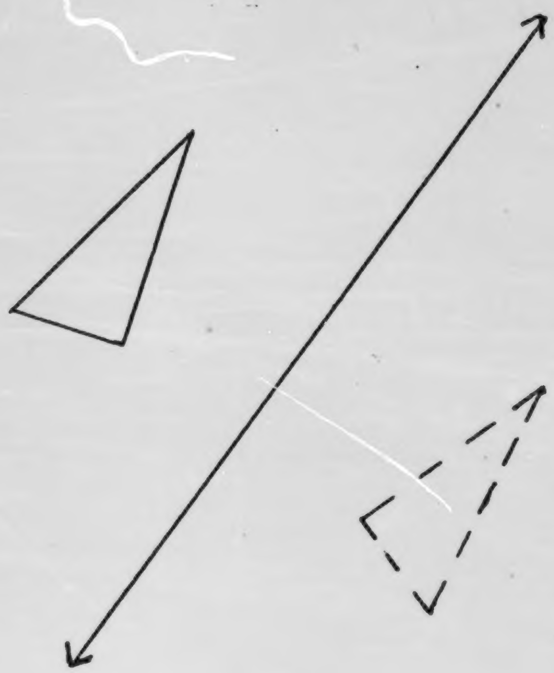

Figure 1. Prototypical flip.

The data did not support Kidder's hypothesis. The majority of subjects in each age group were unable to perform transformations, inverse transformations, and compositions of transformations at the representational level. Failure to conserve length (i.e., the subject did not construct a triangle congruent to the original) was the most prevalent cause for non-performance of the transformational tasks. Sixty-seven percent of all errors committed were failures to conserve length; 71%, 53%, and 67% of the individual motion, inverse motion, and composition errors were failures to conserve length. Furthermore, only four of the 72 subjects attempting the tasks were able to perform all 12 individual motion tasks with no conservation of length errors.

Research supports the proposition that children, ages six to eight, do conserve length in the classical sense. Why, then, was this capability nonevidenced by 9,11, and 13 year old adolescents while performing Euclidean transformations at the representational level? Other studies, some of which are not specifically concerned with conservation of length per se, shed light on factors which may very well effect the child's conservation of length capabilities. Baker and Sullivan (1970) question that there is a well-delineated dichotomy between non-conservation and conservation. Their examination of children's ability to conserve numerousness suggests that such task variables as interest in task object (candy versus checkers) and size of aggregate (4-9) may be factors in a child's response to conservation questions. Musick (1976) reported an investigation of the effects of applying energy (work) and differing speed on the conservation of distance by young children. She found that young children think that the distance across a room is further if they have to carry a sack of groceries as opposed to walking across the room empty-handed, and the distance was thought to vary depending upon whether they walked or ran across the room. During the same workshop, Schultz (1976) reported her study on the effects of task variability upon the child's ability to perform Euclidean transformations. She found that when the gross distance between the original and the image figure was increased to such an extent (distance across the room) that the image figure could not be seen and compared in close juxtaposition with the original, the children's ability to place the image figure in correct position was usually markedly decreased. To test her hypothesis that the global configuration of the figure to be transformed is a factor in the child's ability to perform transformations, Schultz used a simple sailing ship composed of three different colored parts. In general, she found that children could transform the sailing ship and place its image in correct position, one part to the other (i.e., looking like the original sailing ship), much more easily than they could transform the same three colored parts of the sailing ship when arranged in nonsensical manner.* Keller and Hunter (1973) also examined task variability on conservation and transitive problems and found significant differences due to task variability. Shantz and Siegel's (1967) conclusion that conservation may be considered as the child's increasing ability to distinguish between relevant and irrelevant attributes has been referenced previously. Gelman (1969) also considers conservation acquisition as a problem of learning to attend to relevant attributes. And Pratoomraj and Johnson (1966) established that even the type of questions used by the experimenter can be a factor in the answers elicited from children during conservation tasks.

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* Details and specific results are in "Task Variables Influencing the Difficulty of Geometric Transformations for 6 through 10 year olds." Dissertation by Karen A. Schultz, Northwestern University, 1976.

In general, the foregoing suggest that a child's length conservation capabilities depend upon many factors including physical, such as arrangement of configuration or distance between the original and image figure, and cognitive, such as which attribute of the task the child deems important. Lesh (1975) reiterated these factors and added a new dimension--the transformation.* He suggests that not only the figure to be transformed, but also the complexity of the transformation itself effect the child's ability to perform the transformation. Kidder (1976) also noted this when contending that there are two most probable reasons why 6-to-8 year old youngsters are found to conserve length, and yet in his study, 9, 11, and 13 year old adolescents did not conserve length while attempting Euclidean transformations at the representational level (see footnote p.144). One is due to the classical definition where the child's acquisition of length conservation is established by a simple comparison between the stick that was moved and the stick that was not moved. In contrast, in Kidder's study length had to be conserved with a more complex point-set planar figure under more complicated transformations. And very importantly, transformations were at the representational level--no comparison of corresponding sides could be made. A child had to be aware of the necessity to hold length invariant at the outset, and he had to select the proper length sticks with which to construct the image figure prior to making the transformations. Second, it is possible that the ability to perform Euclidean transformations at the representational level derives from formal operational thought (in the Piagetian sense). One general property of formal operational thought is the ability to separate out factors not given by direct observation and the ability to hold one factor constant while observing the causal action of another (Flavell, 1963). Performing Euclidean transformations at the representational level is one such multi-factored mental operation (Kidder, 1976, p. 51). Using Euclidean transformations as a vehicle to test the child's conservation of length therefore increases the possibility that the child will focus attention on one attribute of the task while forgetting or ignoring others.

The preceding theoretical discussion strongly suggests that a child's ability to conserve length is a function of the mental operation involved. Hence, it is hypothesized that conservation of length in the classical sense is a necessary but not sufficient condition for conservation of length while performing simple Euclidean transformations at the quasi-representational level. (See p. 150)

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* The reader is cautioned that the word transformation is given two different meanings in this paper. Here it is used as would the Genevans, with transformations being considered mental operations of the child. Elsewhere, transformations are restricted to their mathematical meaning--being either Euclidean, similarity, affine, etc.

Procedures

The Sample

The sample consisted of 60 subjects--10 each, randomly selected from grades 2, 3, and 4 in the Campus Laboratory School, Longwood College, Farmville, Virginia, and Dillwyn, Virginia Public School System. Mean ages by grade as of April 1, 1976, were 8.2, 9.0, and 9.9 respectively. In selecting grade level, due consideration was given to Jean Piaget's theory of cognitive development.

Administration

The study was patterned after the protocols of Jean Piaget and his collaborators (Piaget, Inhelder, & Szeminska, 1960) in that testing consisted of personal interviews, conducted on a one-to-one basis, with a record being kept of both the child's verbal answers and action responses. The "classical" conservation of length test was administered first. To be sure that the subjects understood what was to be expected of them during the transformation test, each subject was then given an operational definition of the basic Euclidean transformations: translations (slides), reflections (flips), and rotations (turns). Immediately following the operational definition activities, subjects were administered the transformation test.

Classical Length Conservation Test. The classical conservation test used is described on page 143. If a subject conserved length when the right hand stick was moved forward approximately 1 cm, he was retested by placing the right hand stick in the middle of, and perpendicular to, the left one. If he failed to conserve length, the test was repeated, sliding the left-hand stick. If a subject was found to be a nonconserver in the classical sense, and yet he conserved length while performing nine or more of the 12 transformational tasks, he was retested in the classical sense following the transformational test.

Operational Definition. Using wire models of the motion indicators--slide-arrow, flip-line, and turn-arrow--the three motions were operationally defined thus: (a) an original figure (a 1/8 inch square stick approximately four inches long) and a motion indicator were placed in front of the subject, (b) a copy was placed on top of the original, and (c) the motion was demonstrated with the copy leaving the original fixed. To aid the subject in recognizing that the image was the same length as the original, these phrases were used during the verbal instructions: "This is an 'exact' copy of the original. This is how the figure looks to you before you slide (flip/turn) it." After a motion was demonstrated several times with various orientations, the subject was asked to use the copy and perform the indicated motion. A subject was considered to use the copy and perform the indicated motion. A subject was considered operational on a given motion if he performed any 3 of 6 predetermined motions. Fifty-three, 51, and 52 of the 60 subjects performed the slides, flips, and turns shown in figure 2. If a subject was found to be nonoperational on all three motions, he was deleted from the study and an alternate was used (only

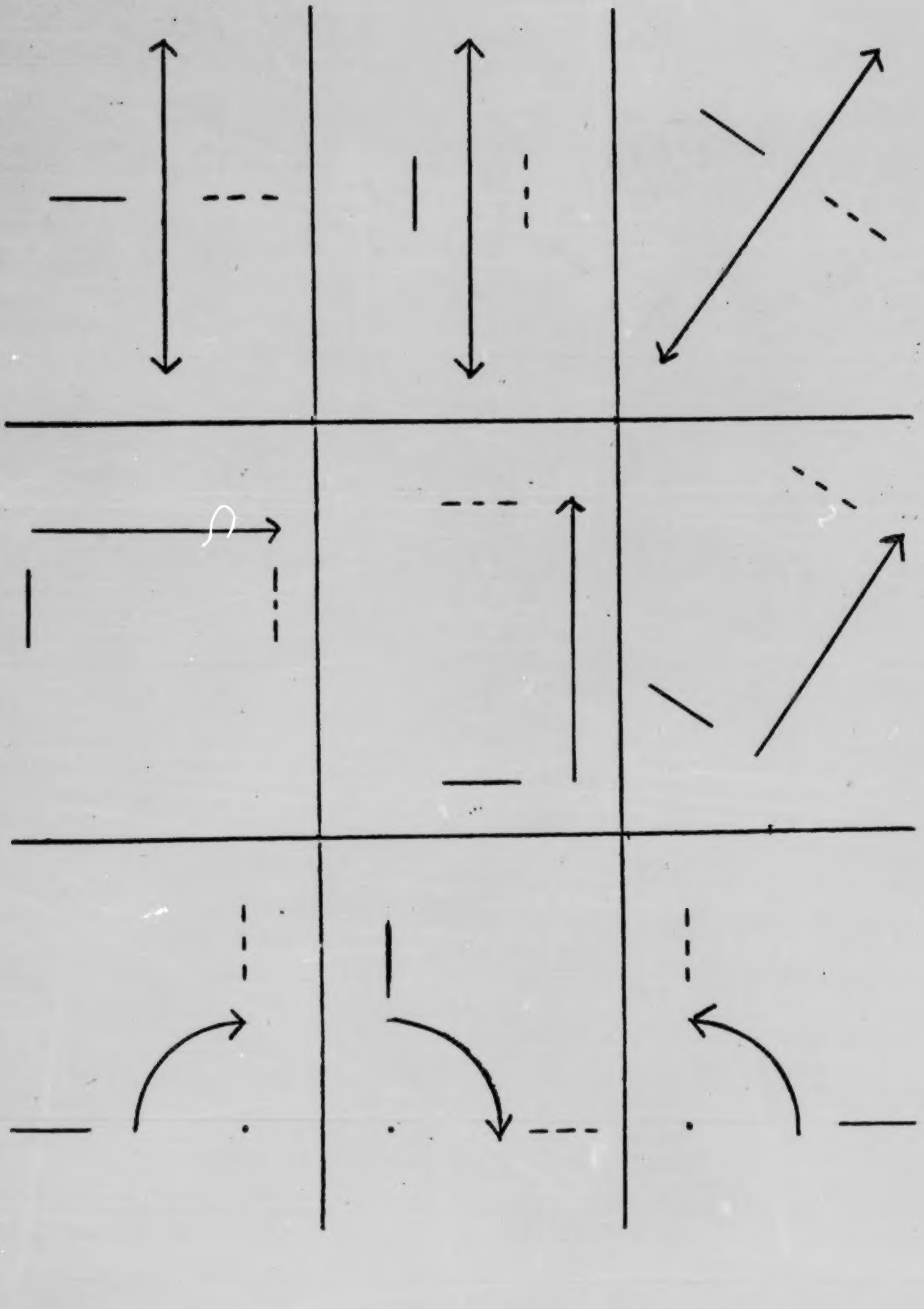

Figure 2. Transformations used in "operational definition".

two subjects--one from the second and one from the third grade--had to be deleted due to lack of understanding).

Transformational Test. The transformational test consisted of 12 tasks, four each of slides, flips, and turns. In each task, the subject was given an original figure (a 1/8 inch square stick approximately 5 inches long), an indicated motion, and five little sticks--one the same length as the original and four of differing length as distractors. He was then asked to use one of the little sticks and show how the original would look to him after performing the indicated motion. The motions exemplified in figure 2 are typical of the motions used in the transformation test; the major difference being the necessity to select the image stick prior to making the transformation. Prior to the first task, to insure that subjects were aware that they could compare lengths of sticks, they were told that they could measure if they wanted to. The motion image was judged to be in one of three categories: (1) correct, (2) image in correct position but failed to conserve length, and (3) conserved length but failed to place image in correct position. After completion of the test, each subject was asked to explain his actions through the questions: "In placing the little stick as it looked to you after being slid (flipped/turned), what were you thinking of? Why did you select the little stick that you did? Did it make any difference which stick you used?"

Rationale. Ideally, in devising an instrument to test the hypothesis that length conservation in the classical sense does not ensure length conservation during a more complex mental operation, the difficulty of only one factor would be increased. In practice this ideal was unobtainable. Preliminary pilot mini-studies indicated that a modification of the classical test by moving the sticks further apart, and/or at a different angle, was not sufficiently more complex to test the hypothesis. The experimenter was still performing all the action, and he was centering the subject's attention on length conservation through his questions. It appears that the experimental task itself has to be one in which the child performs the action and must be such that the subject has to be consciously aware of the need to conserve length without having it brought to his attention through the experimenter's questions. As suggested by Steffe and Carey (1972), any length preserving transformation will suffice if it is such that young children can reasonably be expected to perform it. Hence, slide, flip, and turn tasks similar to the classical test were devised using little sticks. The tasks are considered to be quasi-representational due to the necessity of having to imagine the end-state of the transformation and recognize the invariance of length prior to making the indicated motion. In using these simple Euclidean transformations, the purpose was not to determine if young children could perform transformations, but to use these motions as a vehicle to test their length conserving capabilities.

Previous experience dictated that children in grades 2-4 be used as the sample. Due to the simplicity of the tasks, older children could be expected to perform both the classical test and the transformational tasks; younger children would probably not conserve length in the classical sense.

Results

Thirty-one of the 60 subjects comprising the sample were found to conserve length in the classical sense. The number and percent by grade level is given in table 1. The large number of classical nonconservers is somewhat surprising in view of the mean ages by grade of the subjects (see sample).

Table 1

Number and Percent by Grade Level of Subjects Conserving Length in the Classical Sense

	Grade 2	Grade 3	Grade 4	Total
Number	8	11	12	31
Percent	40%	55%	60%	52%

All of the participants (two alternates were included) demonstrated satisfactory understanding of slides, flips, and turns on the "Operational Definition" activities and were considered operational on all three motions.

As part of the experimental design, conservation of length criteria of nine or more correct length selections was established as consistent conservation on the 12 tasks comprising the transformational test. A subject was considered to be a consistent nonconserver of length on these tasks if he selected the correct length image stick four or fewer times. From 5 to 8 correct length responses was not considered indicative of either consistent length conservation or consistent nonconservation. Table 2 summarizes the performance of the subjects who conserved length in the classical sense.

Table 2

Transformational Test Performance by Subjects Conserving Length in the Classical Sense

Number and Percent of Subjects Conserving Length on Transformational Tasks

Number of Tasks in Which Length Conserved	Grade 2		Grade 3		Grade 4		Total	
	Ss	%	Ss	%	Ss	%	Ss	%
No. $\leq$ 4	5	62.5	7	63.6	7	58.3	19	61.3
5 $\leq$ No. $\leq$ 8	0	0.0	4	36.4	1	8.4	5	16.1
9 $\leq$ No.	3	37.5	0	0.0	4	33.3	7	22.6
Total	8	100	11	100	12	100	31	100

It is seen that the majority of classical conservers failed to conserve length while performing simple Euclidean transformations at the quasi-representational level. Only seven (22.6%) of the 31 classical conservers consistently selected the correct length image stick while performing the transformations, and 19 (61.3%) consistently failed to select the correct length image stick. In total, 24 (77.4%) of the subjects who conserved length in the classical sense failed to consistently conserve length on the transformational tasks indicating that classical conservation of length is not sufficient to ensure length conservation on more complex mental operations. The performance on the transformational test by those subjects who did not conserve length in the classical sense was similar. Table 3 indicates that only six (20.7%) of the 29 classical nonconservers consistently selected the correct length image stick while performing the transformational tasks, and 18 (62.1%) consistently failed to select the correct length image stick.

Table 3

Transformation Test Performance by Subjects Not Conserving Length in the Classical Sense

Number of Tasks in Which Length Conserved	Grade 2		Grade 3		Grade 4		Total	
	Ss	%	Ss	%	Ss	%	Ss	%
No. ≤ 4	9	75.0	6	66.7	3	37.5	18	62.1
5 ≤ No. ≤ 8	0	0.0	1	11.1	4	50.0	5	17.2
9 ≤ No.	3	25.0	2	22.2	1	12.5	6	20.7
Total	12	100	9	100	8	100	29	100

It is to be expected that the performance of the two groups would differ more drastically. The most probable reason why this did not occur is given in the "Discussion" section.

As discussed under "Rationale," the transformational tasks were designed to be slightly more complex than a simple slide of one stick followed by a comparison of their length as in the classical test. Yet, if the tasks were too difficult, the purpose of the experiment would be defeated in that young children would not be able to understand the "Operational Definition" of the motions involved. This did not occur. Table 4 reveals that 61.3%, 65.0% and 74.2% of the 240 tasks were satisfactorily performed as to position of image by subjects in grades two, three, and four respectively.

Table 4

Number and Percent of Transformational Tasks Satisfactorily Performed (as Far as Position) by Subjects

	Grade	S	F	T	Total	No. Tasks	Percent
Classical Conservers	2	12	18	28	58	96	69.0
	3	16	33	36	85	132	64.4
	4	26	34	44	104	144	72.2
Classical Non-Conservers	2	20	28	41	89	144	61.8
	3	14	23	34	71	108	65.7
	4	22	20	32	74	96	77.1
Total	2	32	46	69	147	240	61.3
	3	30	56	70	156	240	65.0
	4	48	54	76	178	240	74.2
		110	156	215	481	720	66.8

Even though the ability to perform transformations at the quasi-representational level was not at issue, it appears that if the original and image figures are reasonably close (in this study they both could be placed on an 18 inch piece of cardboard; also see Schultz, 1976) that young children can visualize or form an anticipatory image of transformations as far as position is concerned. Since length is the essential invariant under Euclidean transformations, and most of the subjects did not conserve length, it cannot be inferred that young children can perform Euclidean transformations at the quasi-representational level. Assuredly, if a subject placed the image stick in correct position but failed to conserve the length of the stick, he was not performing a Euclidean transformation. Yet he was performing a transformation which can be described mathematically. It is tempting to say that the transformation was a "similarity" since the subjects were trying to make their images "like" the original. However, the width of the stick was physically held invariant while the length was permitted to change, making it more appropriate to call the transformation "affine" (see Martin, 1975). Length was not conserved on 427 of the 720 transformational tasks. In attempting these 427 tasks, subjects placed the image stick in correct position 290 (67.9%) times. This suggests that young children may acquire the capability to perform affine transformations prior to the ability to perform Euclidean transformations. Twenty-four, 31, and 45% of the slides, flips, and turns tasks were performed correctly indicating that for these particular tasks, slides were the most difficult to perform, followed by flips and then turns.

Discussion

Most conservation studies have been replications of Jean Piaget's experiments. Few have directly examined the factors involved in a child's ability to conserve, or the suitability of Piaget's tasks to measure this cognitive phenomena. In particular, little attention has been given to the possibility that conservation on Piagetian tasks may not ensure that a child will conserve the same attribute while attempting other mental operations. This study is an attempt to provide answers regarding this possibility on conservation of length tasks.

The data strongly suggests that the ability to conserve length in the classical sense does not ensure length conservation on more complex mental tasks. It could not be definitively determined if the subjects "lost" their ability to conserve length on the more difficult tasks or if they simply "ignored" length as a factor. Even though most subjects were unable to describe their thinking when selecting a stick to place in the image position, questioning revealed that, in general, the subjects who did not conserve length on the transformational tasks were concentrating on where to place the image--ignoring size. In Shantz and Siegel's (1967) terminology, the most relevant attribute of the tasks was the position of the image. In answer to a later, more direct question, "Did it make any difference which stick you used?", 18 of the 29 classical nonconservers and 17 of the 31 classical conservers said that it did not. The experimenter can never be completely sure that he is asking each subject the exact same question with the same emphasis, or just what cues the subject receives from the questions. However, to ensure that the child's conservation/nonoonservation of length on the transformational tasks was not influenced by the experimenter, the questioning was conducted after completion of the testing. The subjects did not retain their capacity to conserve length while performing a more complex mental operation, and it appears that either they were centering on another attribute thereby ignoring the need for length conservation, or slides, flips, and turns are not Euclidean (length preserving) for this age child. Attending to a particular attribute while ignoring others is supportive of the findings of Shantz and Siegel (1967) and Gelman (1969). Making images "like" instead of "congruent to" the original suggests that the child's spatial development, as exemplified in his ability to perform transformations, may indeed proceed from the more global or general notions to the more specific. Martin (1975) hypothesizes that Klein's Erlanger Programm may serve as a geometric model of spaces which a child constructs during development. In Klein's classification, the more general "affine" geometry includes the "similarity" geometry which, in turn, includes the more specific "Euclidean" geometry. Since transformations and invariants under these transformations determine the geometry, selecting a "like" image instead of a "congruent" one during the transformational tasks suggests that there may be merit in Martin's hypothesis, and that a child's spatial concepts may very well be "affine" or "similar" prior to becoming Euclidean.

It is noted that of the 29 subjects who did not conserve length in the classical sense, 18 (62.1%) also consistently failed to conserve length on the transformational tasks and five (17.2%) were indeterminate. That is, a total of 23 (79.3%) classical nonconservers failed to consistently conserve length during the transformational test. This indicates that conservation of length in the classical sense is a necessary condition for length conservation on more complicated tasks. Six (20.7%) of the 29 classical nonconservers did consistently select the correct length stick during the transformational test. These were retested in the classical sense, and all six again failed to conserve length on the Piagetian task. Observation of subjects' actions while performing the transformational tasks indicates that this apparent inconsistency resulted from the training received during the "operational definition" activities. During this instruction the experimenter demonstrated by word and action that an "exact" copy was to be placed on top of the original and that the copy was to be slid (flipped/turned) by the subject, leaving the original fixed. The six classical nonconservers who consistently selected the correct length image stick on the transformational tasks followed these instructions to the letter. They placed a stick on top of the original and if it didn't fit, they tried another until it did. This technique was also observed with other subjects (both classical conserver and nonconserver), and it is the experimenter's opinion that if the transformations could have been operationally defined in a manner less suggestive than placing a copy on top of the original, more of the subjects (both classical conservers and nonconservers) would not have selected the correct length image stick while performing the transformations. It appears that the design of the experiment errs in this respect; however, this strengthens the conclusions that conservation of length in the classical sense is a necessary but not sufficient condition for length conservation during more complex mental operations in that the majority of both classical conservers and nonconservers did not attend to length conservation on the transformational tasks even though the training activities were strongly suggestive that this was important.

There is ample evidence that attempts have been, and are being, made to apply Piagetian theory directly in the classroom. This is so even though Piaget discourages such practice. Notwithstanding, it is highly desirable that mathematics educators make realistic inferences from both Piagetian and related research. The ability or lack of ability of young children to conserve various attributes is well established. One purpose of this study, then, is to highlight the care which must be used if inferences are to be drawn from the expression, "The child conserves length." Just what does this statement mean? The results of this study appear to indicate that it has different meanings in different contexts; that is, the child's ability to conserve length depends upon the mental operation he is performing.

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An Investigation into the Effect of Instruction on the Acquisition of Transformation Geometry Concepts in First Grade Children and Subsequent Transfer to General Spatial Ability

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According to Jean Piaget, the gradual mastery of invariant properties under progressively more complex systems of transformations is the essence of cognitive growth (Piaget & Inhelder, 1971). Could this mastery of simple spatial transformations be the goal of geometry instruction at the elementary level?

If a child rides a tricycle downhill, he is aware that it changes in position, but not size or shape; if his tricycle is hit and damaged, he observes a change different from the first; if he sees someone riding the tricycle in the distance he notices it appears smaller than before. These examples illustrate respectively three types of spatial transformations the child experiences; rigid, topological, and projective. Although the child is aware of these changes, it is only when he is able to organize his experiences, perhaps through instruction, that he develops the necessary perceptive and imaginative schema, as well as vocabulary, to understand the spatial concepts involved.

Currently, there is disagreement among learning theorists and mathematics educators whether instruction can accelearate cognitive development. Unfortunately, there is insufficient research to strongly support either side of the controversy. Similarly, little research has been done to investigate whether knowledge of transformation geometry concepts enhances a young child's general spatial ability. The purpose of this study is two-fold: first, to investigate the effect of instruction on the acquisition of certain transformation geometry concepts in first grade children, and second, to study whether such concepts contribute to a child's general spatial ability. The study was completed as a doctoral dissertation at Northwestern University.

Review of the Literature

Jean Piaget has awakened mathematics educators to the possibility of definite stages in the cognitive development of a child's spatial ability. Throughout his experimentation, Piaget's primary concern has been to study the operational structures involved in mathematical ideas. Piaget's approach is to decide what relations and operations are involved in a mathematical idea and then to investigate how the child structures these cognitively. Piaget has not been particularly concerned about devising instructional materials to accelerate cognitive growth, but he does believe that instruction should be formulated to fit the learner's cognitive structures.

In spite of the fact that Piaget has made few specific prescriptions for instruction, "Piagetian" curriculums and materials have been developed based on his theory. However, as Steffe (1973) suggests, care must be taken in interpreting the theory of Piaget to mathematics instruction.

> ...extrapolating his (Piaget's) results to mathematics education is fraught with unexpected difficulties. These difficulties are of a two-fold nature. The first concerns the relationships between the mathematical-like content of Piaget's theories and the structure of the mathematical content to which the extrapolations are made, and the second concerns the type of data available on which to base the extrapolations.

It was not, of course, Piaget's intent to use the classroom with all of its variables as his laboratory. Nevertheless, there is a definite need for this type of research. Soviet researchers in mathematics education are particularly concerned with the classroom as laboratory. In the introduction to Soviet Studies in the Psychology of Learning and Teaching Mathematics, Kilpatrick and Wirszup (1972) comment:

> ...They (Soviet psychologists) contend that,...spatial concepts and spatial abilities are best studied as they develop under the influence of school instruction. Shunning, like Piaget, psychometric techniques such as factor analysis, Soviet researchers have taken the classroom as the laboratory for studying spatial abilities, and the teacher--working with materials--as the agent for inducing change. In the four selections in the present volume, one sees examples of how spatial concepts are treated in Soviet classrooms, how students' drawings are analyzed to yield evidence about their thinking, and how conclusions from the analysis of behavior are used to guide geometry instruction.

Piaget's experimentation (1971) has led him to conclude that children are unable to comprehend reflections (flips) until age eight and rotations (turns) until age nine. However, many believe that these transformations can be understood to some degree earlier. For example, both Shah (1968) and Williford (1972) have developed teaching studies in transformation geometry for primary school children which indicate that even first graders are able to acquire some basic transformation concepts. Nonetheless, more research is needed to show whether a knowledge of transformation geometry contributes to one's general spatial ability. Shah's study did not address this question, and Williford's study produced negative results; i.e., the experimental group did not score significantly higher than the control group on the post-instructional space test. The items for the space test were taken from several widely used IQ tests for primary school children.

Sample

The sample consisted of 72 first grade students of average and above average ability from two public elementary schools in northeast Chicago, Rogers Elementary School and Field Elementary School. The two schools drew students from neighboring communities with similar populations. So, the schools were roughly equivalent in ethnic balance of socio-economic status of their children. For instructional purposes, the experimental group was divided into three classes of 12 students each, two classes at Rogers and one class at Field. Similarly, there were three control classes, two at Rogers and one at Field.

Since I.Q. tests are not administered to first graders in Chicago public schools, students were ranked according to ability by their individual teachers. Random procedures were used to assign students to the three experimental and three control groups so that the groups would be equivalent in ability.

Instructional Unit

The instructional unit was administered in 11 sessions of 30 to 35 minutes each. The first session was given to both the experimental and control classes to familiarize both groups, particularly the control groups, with the terminology used in the achievement test. Two-dimensional cardboard figures were used to give an overview of the concepts of slide, flip, and turn. Two classes were held with the experimental groups to develop the concept of a slide, four classes on the concept of a flip, and four classes on the concept of a turn. More class time was spent on the latter two transformations since studies by Shah (1968), Williford (1972) and Moyer (1973) indicate that slides are easiest for first graders to comprehend. The modes of instruction included both lecture-discussion and small group work. The unit activities may be classified in the following way:

Transformation	Classification	Activity
slide, flip	horizontal	
	vertical	tracing paper
	diagonal	geoboard
	internal*	free drawing
turn	45°	tracing paper
	90°	geoboard
	180°	free drawing

A complete description of the instructional materials that were used are presented in Perham (1976).

Instrumentation

An achievement test of 80 items involving transformations, 28 multiple choice items and 52 dichotomous items, and a multiple choice spatial ability test of 23 items was given as a pre-unit and post-unit test to all classes. A complete copy of this test is given in Perham (1976). The multiple choice items in the transformation test evaluated the child's ability to anticipate the final image of a transformation; the dichotomous items tested the child's representation (drawings) of various transformations. Each drawing was evaluated on the basis of four criteria: size, orientation, use of the referent given, and angle conservation. The 23 spatial ability items were taken from currently used spatial ability tests of three major companies: Educational Testing Service, Psychological Corporation, and Science Research Associates. These items evaluated five areas of spatial ability: completion of shapes, left-right orientation, perspective, figure folding, and reasoning.

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* Internal flips are those for which the line of reflection is internal to the given figure, either horizontal or vertical.

Statistical Design

An item analysis was performed on all multiple choice and dichotomous items of the achievement test as a whole as well as each subtest. Pearson point biserial correlation coefficients were computed for dichotomous items to show both item-test and test-test correlations. A homogeneity analysis was performed on both multiple choice and dichotomous items; a Cronbach-S alpha index was computed for the test as a whole and for each subtest. These reliability indices give an average of all correlations between a given subtest and all possible combinations of remaining test items.*

Gain scores of the treatment group on the transformation test and all subtests as well as the space test were computed. Comparisons of mean gain scores of the treatment group with the control group were made using a t-statistic. This statistical design was chosen to investigate the following hypotheses:

(1) There will be no significant difference between the pre-test scores of the experimental group and the pre-test mean scores of the control group on any sub-tests of the transformation test.

(2) There will be no significant difference between the pre-test mean scores of the experimental and control groups on any sub-tests of the spatial ability test (multiple choice items).

(3) There will be a significant difference between the mean gain scores of the experimental and control groups on all subjects of the transformation test.

(4) There will be a significant difference between mean gain scores of the experimental and control groups on the sub-tests of the spatial ability test.

Results

The item analysis on the transformation test yielded significant Pearson point biserial correlation coefficients for most item-test and test-test correlations. The Cronbach-S alpha indices for the test as a whole and each sub-test were also high.

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* See Perham (1976) for complete statistical analysis.

Significant t-statistics for the transformation test and spatial ability test are given in tables 1,2, and 3, below.

Table 1

Significant t-Statistics for Transformation Test

Multiple Choice Items

Variable	Group	Mean		Mean Gain	% Gain	t-statistic	
		Pre-test	Post-test			Pre-test	Mean gain
Flips							
Horizontal	E	1.81	2.75	0.94	23.5	0.95	2.76**
	C	1.56	1.31	0.25	-6.3		
Vertical	E	1.64	2.56	0.92	23.0	0.32	2.02*
	C	1.58	1.56	0.02	-0.5		
Diagonal	E	1.17	1.81	0.64	16.0	0.43	0.71
	C	1.08	1.22	0.14	3.3		
Internal	E	1.33	2.00	0.67	16.8	1.21	1.81*
	C	1.53	1.72	0.19	4.8		
All flips	E	5.94	8.03	2.09	17.4	0.41	3.48**
	C	5.75	6.00	0.25	1.6		
Turns							
45°	E	2.19	3.17	0.98	24.5	0.80	1.90*
	C	2.08	1.86	0.22	-5.5		
90°	E	2.11	2.75	0.64	16.0	0.44	1.90*
	C	1.92	2.19	0.27	6.8		
180°	E	2.61	3.00	0.39	9.8	0.73	1.70*
	C	2.39	2.19	0.27	-5.0		
All turns	E	6.92	8.92	2.00	16.7	0.30	3.15**
	C	6.25	6.25	0.00	0.0		

* $p < 0.05$.
** $p < 0.01$.

Table 2

Significant t-Statistics for Transformation Test Drawings

Variable	Group	Mean		Mean Gain	% Gain	t-statistic	
		Pre-test	Post-test			Pre-test	Mean gain
Slides							
Horizontal	E	3.08	3.17	0.09	2.0	0.84	0.11***
	C	2.92	2.86	0.06	1.4		
Vertical	E	2.70	2.74	0.04	1.0	0.13	0.14***
	C	2.65	2.69	0.04	1.0		
Diagonal	E	2.31	2.36	0.05	1.3	0.22	1.30
	C	2.28	2.06	0.22	5.5		
Internal	E	2.08	2.22	0.14	3.5	0.11	0.95
	C	2.06	1.94	0.12	2.8		
All slides	E	10.08	10.36	0.28	1.4	0.41	0.70
	C	9.64	9.28	0.36	2.2		
Size	E	1.22	1.67	0.45	11.2	1.29	2.39*
	C	0.92	1.22	0.30	7.6		
Orientation	E	3.36	3.91	0.55	13.7	0.36	1.75***
	C	3.17	3.03	0.14	3.4		
Angle	E	2.75	3.81	1.06	26.5	0.19	0.19
	C	2.81	2.67	0.14	3.3		
Referent	E	2.78	3.00	0.22	5.5	0.11	0.74
	C	2.75	2.78	0.03	0.8		
Flips							
Horizontal	E	2.11	3.42	1.31	16.4	1.45	2.86**
	C	1.71	2.61	0.90	11.2		
Vertical	E	1.86	3.00	1.14	14.2	1.31	2.50**
	C	0.92	2.33	1.41	17.6		
Diagonal	E	2.08	2.56	0.48	6.0	1.06	1.63
	C	1.56	2.19	0.63	7.9		
All flips	E	4.18	8.98	4.80	20.0	-0.26	4.76**
	C	6.05	7.13	1.08	4.5		
Size	E	1.42	1.75	0.33	5.5	0.26	3.49**
	C	1.31	1.31	0.00	0.0		
Orientation	E	2.17	2.92	0.75	12.5	0.48	2.82**
	C	2.06	1.72	0.34	5.7		
Angle	E	3.89	4.25	0.36	6.0	0.83	3.92**
	C	3.67	3.36	0.31	5.2		
Referent	E	2.39	2.92	0.53	8.8	0.07	3.84**
	C	1.72	1.86	0.14	2.3		

Table 2--Continued

Variable	Group	Mean		Mean Gain	% Gain	t-statistic	
		Pre-test	Post-test			Pre-test	Mean gain
Turns							
45°	E	1.28	3.17	1.89	47.3	1.25	0.83
	C	2.61	1.86	0.75	18.7		
90°	E	1.56	2.75	1.19	29.7	0.69	0.92
	C	2.00	2.19	0.19	4.7		
180°	E	1.00	3.00	2.00	50.0	0.08	1.20
	C	1.30	2.19	0.89	22.2		
All turns	E	3.83	8.92	5.09	42.4	-0.71	1.65
	C	5.97	6.25	0.28	6.9		
Size	E	1.17	1.22	0.05	1.3	-0.48	0.13
	C	0.92	1.22	0.30	7.6		
Orientation	E	0.97	1.36	0.39	9.7	-1.00	1.69
	C	1.17	1.36	0.19	4.7		
Angle	E	2.06	2.00	0.06	1.3	1.29	1.05
	C	1.72	2.00	0.28	7.0		
Referent	E	1.89	1.78	0.11	2.7	1.20	-0.63
	C	1.83	1.78	0.05	1.3		

* $p < 0.05$.
** $p < 0.01$.
*** High pre-test and post-test means.

Table 3

Significant t-Statistics for Special Ability Test
Multiple Choice Items

Variable	Group	Mean		Mean Gain	% Gain	t-statistic	
		Pre-test	Post-test			Pre-test	Mean gain
Completion of shapes	E	3.13	3.69	0.56	11.2	0.74	0.54***
	C	3.61	3.72	0.11	-2.2		
Left-right orientation	E	2.69	2.78	0.09	3.0	1.01	0.71***
	C	2.50	2.53	0.03	1.0		
Perspective	E	3.69	4.67	0.98	19.6	0.09	3.23**
	C	3.58	3.67	0.09	1.8		
Figure folding	E	1.84	1.39	-0.45	-9.0	0.65	-0.11
	C	1.53	1.25	-0.28	-5.6		
Reasoning	E	2.00	2.06	0.06	1.2	0.38	1.24
	C	1.78	1.53	-0.25	-5.0		
All items	E	13.35	15.02	1.67	7.2	0.61	0.70
	C	13.05	12.69	-0.36	1.5		

** $p < 0.01$
*** High pre-test and post-test means.

These results may be summarized as follows:

(1) As expected, the experimental group did not score significantly higher on any pre-unit transformation or spatial ability sub-test.

(2) The experimental group scored significantly higher than the control group on: horizontal, vertical, and internal flip multiple choice post-unit sub-tests and the post-unit flip test as a whole; 45°, 90°, 180° turn post-unit sub-tests and the post-unit turn test as a whole; horizontal and vertical flip post-unit drawing test and the post-unit flip drawing test as a whole; perspective (reflection) post-unit spatial ability subtest.

(3) The experimental group did not score significantly higher than the control group on: diagonal slide post-unit sub-test and diagonal flip post-unit subtest; 45°, 90°, 180° turn drawing post-unit sub-tests or the turn drawing post-unit test as a whole; any slide post-unit sub-test, although mean scores were high for both experimental and control groups on the horizontal and vertical subtests; completion of shapes, left-right orientation, figure folding, or reasoning post-unit sub-tests. The mean scores of the experimental and control groups are high for the completion of shapes and left-right orientation sub-tests.

Conclusions

The high mean pre-unit test scores for horizontal and vertical slides suggest that prior to instruction the first graders understood translations at the anticipatory and representational levels.* There was no significant difference between mean gain scores of the two groups for diagonal slides, suggesting that children at this level are not able to learn diagonal translations. Also, there was no significance for diagonal flips, although there was significance for every other type of flip--horizontal, vertical, and internal. Although there was significance for mean gain scores for all types of turns at the anticipatory level, there was no significance for any type of turn at the representational level.

Overall, these results reaffirm both the existence and the order of Piaget's proposed levels of cognitive growth in understanding transformation geometry concepts (Piaget & Inhelder, 1971). First grade children prior to instruction had some understanding of both horizontal and vertical translations (slides) at both the anticipatory and representational levels, but did not comprehend relections (flips) or rotations (turns) of any type at either of the two levels. After instruction, the experimental group of children understood not only horizontal and vertical slides, but horizontal, vertical, and internal flips at both the anticipatory and representational levels. Instruction was some what effective in increasing the understanding of 45, 90, and 180 degree turns but only at the anticipatory level. These results are consistent with Piaget's conclusions that children learn transformations in the order slides, flips, and turns, and that the anticipatory level precedes the representational level of understanding.

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* Anticipatory understanding was evaluated by the multiple choice items; whereas, represeniional understanding was assessed by the dichotomous items requiring the construction of the transformation image.

An important result of this investigation is that after instruction the first graders scored significantly higher on all types of flips (except diagonal) at both the anticipatory and representational levels. However, this does not necessarily contradict Piaget's findings that children are not able to comprehend flips until approximately age eight or nine since Piaget's experimentation did not exhaust all the possibilities for degrees of complexities of figures used, the types of transformations, and the ways in which the transformations were performed. These results do suggest that although children may not be able to comprehend every type of reflection, they are able to learn some types, namely those involving simple figures where the line of reflection is horizontal or vertical.

What is even a more significant result of this study is that an understanding of transformations performed diagonally (including slides) was not achieved after instruction. This contributes a new dimension to the understanding of cognition; i.e., transformations that are not performed horizontally or vertically (including rotations) are difficult for first grade children to understand. Perhaps this is because such transformations require a re-orientation of the horizontal-vertical coordinate system with which the child is familiar. Nevertheless, these results do indicate that the categorization of rigid transformations into slides, flips, and turns may not be as meaningful for instruction as one by orientation, i.e., horizontal, vertical, diagonal, etc.

A major question was the relationship of transformation geometry concepts to general spatial ability. However, the spatial ability test did not provide conclusive results, due to a large difference in performance on various subtests. Prior to instruction, the first grade children scored relatively high on the recognition of shapes and left-right orientation sub-tests, relatively low on perspective, and very low on figure folding and reasoning. The only significant change after instruction was that the experimental group scored significantly higher on the perspective sub-test. Before considering the dramatic change on the perspective sub-test, let us first look at the ceiling effect on the first two sub-tests and the low scoring on the latter two sub-tests.

The fact that the first graders scored extremely well on the pretest of the recognition of shapes and left-right orientation indicates that the constructors of such spatial ability tests might well analyze if the said tests are sufficiently discriminating or not. Given that children are exposed to a number of educational television programs prior to entering first grade, pre-school children might have been exposed to recognition of circles, triangles, squares, etc., to the point of overkill. On the other hand, the fact that the children scored so low on the figure folding and reasoning subtest may indicate that such tests are indeed too difficult and need to be re-examined.

However, the prespective subtest did provide a possible link between transformation concepts and spatial ability. The significant difference between mean gain scores for this subtest combined with the fact that the most significant improvement of the experimental group from pre-test to post-test on the transformation geometry achievement test was on the flip (reflection) subtests suggests that an understanding of reflection is directly related to their understanding of perspective. This possible relationship certainly merits further investigation.

Since this study is the first of its kind at the first grade level, it was by necessity broad in scope. Further research needs to be done in a more concentrated fashion for each sub-category tested. For example, an entire teaching study devoted to translations and reflections performed diagonally would be useful. Similarly, a teaching study solely on reflections and subsequent transfer to spatial ability would also be valuable.

The limitations on the length of the test prohibited a larger number of items for each subtest as well as a larger number of subtests. Given this restriction and the limitations of any evaluative instrument for first grade, the results of this inquiry strongly suggest that first graders are able to learn many transformation geometry concepts, not only translations, but reflections and rotations as well.

Those developing the elementary mathematics curriculum would be well advised to include simple transformation tasks in their first grade texts. Horizontal and vertical slides and flips at both the anticipatory and representational levels as well as turns (45°, 90°, and 180°) at the anticipatory level seem to be appropriate for first graders.

The modes of instruction used in this teaching experiment, namely lecture-discussion and small group work, and the unit activities, which included tracing paper and geoboard tasks as well as free drawing, worked very effectively. Students at this age enjoy "hands-on" activities, and furthermore, Piaget and his replicators have shown that learning begins at the concrete level and the less abstract the tasks the better, at least at this level. Hence curriculum builders might consider incorporating similar tasks in the transformation geometry curriculum.

In the introduction it was pointed out that a controversy exists as to whether instruction can to any significant degree accelerate cognitive growth. This investigation indicates that although instruction can accelerate growth to some degree, as demonstrated by the improved understanding of horizontal and vertical flips following instruction, the amount of growth is limited, as indicated by the lack of improved performance on tasks involving diagonal transformations. Hence, this study, as well as that of the Russian and Dutch psychologists and educators, as well as Shah (1968) and Williford (1972) seem to suggest that instruction can to some degree move a child from one level of cognition to another; that we must not interpret the theory of Piaget as a passive one; and that we must expand the research of Piaget to the classroom with all its variables.

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An Analysis of Research Needs in Projective, Affine, and Similarity Geometries, Including An Evaluation of Piaget's Results in These Areas

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Toward the end of The Child's Conception of Space, Piaget states:

> We have seen how the simple topological notions with which the child begins to construct the concept of space were transformed concurrently into projective and Euclidean concepts. The first of these, embracing perspective, sections, projections and plane rotations, results from the co-ordination of viewpoints, while the second derives from the conservation of straight lines, parallels, angles, and lastly, general co-ordinate systems.

Throughout Piaget's space books, The Child's Conception of Space and The Child's Conception of Geometry, Piaget asserts that projective and Euclidean concepts develop concurrently. However, the design of the chapters, the research reported within them, and other comments suggest that after the development of certain basic topological notions, spatial concepts tend to develop from projective (straight lines preserved) to affine (parallels) to similarity (angles) to Euclidean (length). This latter point of view has been analyzed and criticized in a number of articles. For example, Martin (1976b) discusses the Erlanger Programm as a model of the child's construction of space and proposes some research questions arising from the Erlanger Programm. Martin's paper also contains a fairly detailed discussion of the transformations involved in the Erlanger classification of geometries. Robinson (1976) reviews some mathematical history that is relevant to these various geometries; and Lesh (1976) discusses some of the issues involved in the use of transformation geometry, particularly Euclidean transformations, in the elementary school. However, concerning the development of spatial/geometric concepts, most analyses of Piaget's claims have focused on topological concepts or Euclidean concepts. That is, from the point of view of an Erlanger-type development, most analyses have neglected the "middle" geometries: projective, affine, and similarity. This paper will give a brief review of some of the most important past research concerning these middle geometries and will isolate some possible future research issues concerning them.

Some Preliminary Comments About the Erlanger Classification of Geometries

The classification of geometries which has come to be called the Erlanger Programm stems from a paper written by Felix Klein in 1872. Although certain subjects, like "topology", had not yet been developed in anything like their present form, Klein's basic proposal was to consider geometry as a search for invariants under specified groups of transformations - and to classify the various geometries according to the properties that are preserved under progressively more restricted systems of transformations.

One of the problems with using the Erlanger Programm to describe the cognitive development of spatial concepts is that the mathematical organization is itself rather complex and not terribly clear. For example, when one attempts to move from topology to projective geometry, the point at infinity must be added. Nonetheless, a naive interpretation of the basic Erlanger organization remains a helpful intuitive scheme for organizing past research results and for generating future research questions. For example, in this book, Weinzweig's article draws upon Erlanger-type organization of geometry concepts to describe some of children's early spatial concepts.

A second problem with using an Erlanger-type scheme to describe children's spatial concepts is that psychologists and educators do not always use mathematical words in the way mathematicians use them. For example, Piaget's double use of the word projective (when referring to the general co-ordination of viewpoints and also to more restricted concepts in the projective, affine, similarity context) is extraordinarily confusing. In this paper, projective will be used only for the latter meaning. Piaget's choice of the word projective for the former meaning was particularly unfortunate, for it is both too restrictive and too narrow. Co-ordination of viewpoints is related to the growth out of egocentrism in all areas, and thus goes beyond projective concepts. This co-ordination is also more restricted than projective concepts, since most of the concepts involved are projections, not projective concepts (this is discussed later in the section on projective research).

One classification of properties and relationships resulting from this classification of geometries as arising from a nested series of transformations is given in table 1. These properties and relationships will be referred to throughout the paper. It should be noted that each type of geometry has the properties noted for it plus all of the properties of the geometries above it. Three dimensional versions of some of the properties are in parentheses. In this country the geometry taught in schools is largely limited to plane geometry. Because plane transformations are easier to imagine and because two dimensional algebraic expressions are simpler, this planar emphasis continues in coordinate geometry and in the study of vectors. This emphasis on two dimensional geometry is a peculiarly American phenomenon, and it probably leads to errors in research on the development of children's geometric concepts and on the type of geometric topics that are taught to children. Children have a great deal of experience with three dimensional space that can be utilized in learning. So, the impact of this experience on the development of geometric ideas should be systematically researched. For this reason, this paper will consider relations in three dimensions as well as in two.

Table 1: Properties and Relationships that Remain Invariant Under Certain Transformation

Topological Properties:	separation (enclosure) neighborhoods concurrence and intersection of curves (surfaces) openness linear, cyclic order of points continuity
Projective Properties:	collinearity (coplanar) this also involves straightness or rectilinearity number of sides of a polygon (polyhedron)
Affine Properties:	convexity betweeneness parallelism ratios of lengths along the same line (in particular, midpoints are mapped to midpoints) ratios of distances along parallel lines
Similarity Properties:	angles ratios of lengths along nonparallel lines
Euclidean Properties:	length

Research Needs in Projective, Affine, and Similarity Geometries

Many of an infant's and a young child's early experiences involve objects which have projective, affine, or similarity properties. Children must come to an intuitive understanding of these ideas in order to function well in space. Ratio is used to make judgments about the "real" size of objects. Many household objects and toys involve ideas of similarity of shape. Parallelism exerts an influence in perception; edges of tables, walls, etc. are used to organize the visual world. The development of perspective involves projective ideas; perspective is used in the initial organization of the visual world and later in representational drawing. Yet, the influence of all of this experience has been neglected by theory and by research. In particular, the fact that much of this experience is with three dimensional objects has not been systematically considered.

This experience begins at birth and extends throughout the school years. This period encompasses three different theoretical spaces differentiated by Piaget: perceptual space, representational space (which involves images), and conceptual space. "Perception is the knowledge of objects resulting from direct contact with them" (Piaget & Inhelder, 1967, p. 17). A mental image is "the evocation of a model without direct perception of it" (Piaget & Inhelder, 1971, p. 4). Conceptual space results when static, whole, unanalyzed images become subject to operations and become capable of being analyzed and coordinated. Perceptual space undergoes an initial construction in the sensory-motor stage (0-18 months); representational space, in the preoperational (1½-7); and conceptual space, in the concrete-operational stage (7-12). Amplification and modification of this original construction continues throughout life. An analysis of relationships among perception, imagery, and conception is given in an article by Lesh in this monograph.

Piaget's research on projective, affine, and similarity ideas has been largely in conceptual space. New studies concerning the development of middle geometry ideas in perceptual and representational space would be valuable in themselves, and they might provide considerable information relevant to the development of conceptual space. For example, the research reviewed in Piaget and Inhelder's Mental Imagery in the Child indicates that the relationships among these spaces are not simple. Discovering more about the first two might help to explain anomolies in the third, conceptual space.

There is a large body of research in psychology on perception and on imagery. Before new studies are begun, this research might be examined to see what is already known about the development of projective, affine, and similarity concepts. In particular, the geometry properties and relationships in table 1 might provide foci for such literature searches. The study of a child's acquisition of mathematical concepts by psychologists is frequently marred by over-simple views of the concepts involved. For this reason, the project of gathering and analyzing what the research literature already contains about the development of middle geometry ideas seems to be a particularly good one for researchers with a mathematics background. Readers who wish to gain initial access to this psychological literature on the development of spatial concepts in children will find the following volumes to be helpful entres: Piagetian Research: A Handbook of Recent Studies (Sohan Modgil), Children's Spatial Development (J. Eliot and N. Salkind, Eds.), and Infant Perception: From Sensation to Cognition, Volumes 1 & 2 (L. B. Cohen & P. Salapatek, Eds.). Laurendau and Pinard (1970) raise interesting and helpful methodological problems, and Olson (1970) provides a stimulating theoretical discussion about spatial development.

Studies are needed which trace the development of a single geometry property through perceptual, representational, and conceptual space. Fuson and Murray's study reported in this volume traced the development of simple Euclidean shapes from perception (haptic-visual, in this case) to representation (building shapes with sticks) to conception (drawing shapes). The order of difficulty of the shapes was invariant in each of the spaces. Other studies might trace the development of geometry concepts such as rectilinearity-curvilinearity (planar-not planar), number of sides of a polygon (polyhedron), parallelism, or angles through perceptual, representational, and conceptual space.

Research which examines the development of projective, affine, and similarity ideas within the same children is vitally needed. This is one of the major problems with the research of Piaget--different concepts are studied with different children, so comparisons between the results of different studies can be made only by using the age of the child. In the analysis of Piaget's results which follows, across-task comparisons will be made. But, as will be seen, these comparisons are often so rough that they are not very informative.

A final area of research in which a focus on projective, affine, and similarity concepts might prove valuable is in the systematic exploration of what Piaget calls decalages--the distribution over a period of time of the ability to do operationally isomorphic tasks that differ only in figurative content. This dispersion results from variance in the "resistance" of objects to the given task. For example, a child does not become able to draw all Euclidean shapes at the same time--some of them are more difficult and occur later than others. Some of these differences in "resistance" might well involve projective, affine, or similarity relations (e.g., figures having parallel sides might be simpler than those without such sides). Because differences in figurative content may create problems in mathematics learning, research which determines important characteristics of such differences would contribute to the design of improved mathematics instruction.

An Analysis of Piaget's Research on Projective, Affine, and Similarity Relations with Suggestions for Related Research

Piaget's research in each area will first be outlined and then analyzed. All ages given are average and approximate. In some cases, Piaget specified the ages in his discussion of the stages. In other cases, ages were not given explicitly and were approximated from the theoretical discussion and from the protocols. The research summarized here is from The Child's Conception of Space, pages 153-270 and 303-375.

Projective Concepts

Piaget conducted five different experiments concerning projective concepts. He approached projective ideas mainly by examining a child's use of perspective. For Piaget, a projective idea requires that the child be aware of his viewpoint and be able to relate a particular viewpoint with what is seen from that place.

Experiment 1: The Straight Line

Matchsticks were set vertically in small globs of placticine. A child could move these about individually to form a straight row of sticks. Two of the sticks were placed 20 cm (or 30 or 40 cm) apart. Children were told that these were telegraph poles and that they had to place the other poles between these two so they ran along a straight road. The results were as follows:

Stage 1 (< 4)	the child cannot form a straight line even when it runs parallel to the edge of the table
Stage IIA (4 - 6)	can form roads parallel to the edge of the table but not diagonally across it
Stage IIB (6 - 7)	through successive trial and error, begins to form straight lines anywhere on the table
Stage III (> 7)	a straight line can immediately be constructed anywhere on a table and a child either constructs or checks his construction by sighting along the line

Experiment 2: Objects Seen by a Doll

The second experiment involved asking children what various objects (needle or stick, circle, ellipse, semicircle, etc.) looked like from different positions. A doll was placed at right angles to the child and the object held up in front of the child and the doll. The child was asked what the object looks like to the doll. The object was also moved through 90° or 180°, and the child asked what it looked like at various intermediate stages or what it will look like at the final stage. Children were also asked about two lines meeting at a distance--railroad tracks or two roads. Children were asked to make drawings in response to these questions and also to select the correct drawing from a set of prepared drawings. The results were as follows:

Stage 1 (< 4)	no geometrical drawings at all
Stage IIA (4 - 5.5)	the object is shown with shape and size unvaried whatever its position relative to the observer
Stage IIB (5.5 - 7)	begins to distinguish between different viewpoints; often chooses correct drawings but is unable to make correct drawings

Stage IIIA (7 - 8)	the general shape of drawings is correct, though transitions are sudden rather than gradual--railway lines are parallel for most of the way and suddenly converge to a point, a stick turning towards a child gradually decreases in length and suddenly on the last step becomes a dot
Stage IIIB (8 - 9)	children accurately portray the gradual intermediate stages as well as the correct final stages in their drawings

Experiment 3: Projection of Shadows

The tasks from experiment 2 were repeated (with different children) by projecting shadows of the objects. Various objects were placed between a lamp and a vertical white screen, separated by only a few centimeters. The child was asked to draw or choose the drawing of the shape he expects the shadow to assume. The objects presented were: a cone, a bobbin (two cones placed point to point), a circle, a rectangle, a pencil. Objects were placed at different orientations. The shapes having constant cross-sections (called simple shapes below) could be done at the same stages as they had been in the previous experiment, except that the pencil seen head-on as a small circle was easier with the shadows.

Stage 1 (< 4)	no geometrical drawings at all
Stage IIA (4 - 5.5)	the shadow is predicted as the child sees from where he stands
Stage IIB (5.5 - 7)	begins to distinguish different projections, especially the vertical and horizontal positions; still fails to predict oblique or intermediary stages
Stage IIIA (7 - 8)	correctly chooses and draws end-stages for the simple shapes, but still cannot portray all of the intermediate stages in turning a shape

Stage IIIB (8 - 11)	correctly shows all intermediate stages in turning simple shapes; still makes errors with the conical shapes--conical shapes may be predicted as that which can be seen from the viewpoint of the light (e.g., a point instead of a circle if the tip of the cone is oriented toward the light)
Stage IV (11 - 12)	children realize that a shadow is the product of something which obscures the light and so can correctly predict the conical shapes

Experiment 4: Three Mountains as Seen by a Doll

The fourth projective experiment done by Piaget was the now famous three mountains experiment. A three dimensional pasteboard model one meter square and from 12 to 30 centimeters high was used to represent three mountains separated by valleys. Each mountain was of a different size, a different color, and had a different object placed on its peak. A doll (2-3 centimeters tall) was placed somewhere on the floor of the layout. Three methods of obtaining a child's answer were used. A child was given cardboard replicas of the mountains and asked to place them as they would look if the doll took a snapshot of the mountains where he was standing, or he was aksed to draw such a snapshot. Alternatively, the child had to select from a group of pictures the view that the doll saw.

Stage 1 (< 4)	children do not understanding the meaning of the questions
Stage IIA (4 - 5.5)	each time the doll is moved, a new picture is drawn but each picture is drawn from the point of view of the child observing, not of the doll. A child may select any picture at random, indicating that the child thinks all pictures are suitable from all views, as long as they show three mountains
Stage IIB (5.5 - 7)	a child attempts to select a picture or draw a picture from the doll's point of view, but cannot do this correctly
Stage IIIA (7 - 8.5)	children realize that different viewpoints exist but these are not considered as mutually exclusive; a medley of viewpoints (a juxtaposition of items belonging to different points of view) may be given;

Stage IIIA (cont.)	the whole layout may be rotated 180° to portray an opposite view
Stage IIIB (8 - 9)	accurate portrayal of a given viewpoint; occurs earliest when using the three pieces of cardboard to construct a layout

Experiment 5: Sections of Geometrical Solids

A final experiment of Piaget's which involves projective ideas is that of sections of geometrical solids. Children were asked to predict the shape of the surface produced when various solids made of plasticine were cut across various places. The objects included a cylinder, prism, parallelepiped, hollow square, cone, and irregular shapes such as twists, snail-shells, paper cornets, etc. The children were asked to draw the surface and to select from pictures of the possible surfaces.

Stage I (≤ 4)	different surfaces are not even distinguished from each other
Stage IIA (4 - 6)	a child cannot separate the internal section from the external, and drawings show a jumble of these features; cannot select the correct drawing
Stage IIB (5.5 - 7.5)	the sectional surface is gradually distinguished from the intact shape
Stage IIIA (7.5 - 8.5)	success for the cylinder, prism, parallelepiped, hollow bowl, and conic sections; cannot do the spiral shapes
Stage IIIB (8.5 - 11)	progress in the spiral shapes but not complete success
Stage IV (≥ 11)	success with spiral shapes

Summary of the Projective Experiments

First, Piaget's tasks were projections rather than being projective. His tasks involved the projection of a three dimensional object onto a plane, either by cutting, by a shadow, or by a picture. This plane projection naturally depends upon the point of projection (or the point of view of the viewer). A projection in this sense is a mapping of an n-dimensional space onto an n-1 dimensional space. But the projective transformations that leave invariant the properties outlined in table 1 are transformations of a plane into a plane, or in general, of an n-dimensional space into an n-dimensional space. With the possible exception of his first task (the straight line), Piaget did not really examine projective geometry, in the sense of a geometry that forms a hierarchy involving affine, similarity, and Euclidean geometries.

Even if one accepts Piaget's meaning of "projective"--a conscious awareness of a viewpoint plus the ability to coordinate a particular view with a particular viewpoint, it is obvious that the last part of this definition may require many different experiences and skills. Tasks 2, 3, and 5 involved knowledge about geometric solids, task 3 required some experience with shadows, and task 4 required the ability to coordinate several different objects in two dimensional space. Training studies which equalize the amount of experience and skill in these special areas, or new tasks which do not require these extra skills would give more insight into the development of projections.

Projections are a common experience in the life of a child. Discovering the invariance of a three dimensional object as it is seen from different viewpoints is a major accomplishment. Building up images of three dimensional objects when one can never see them from more than one viewpoint (i.e., from one point of projection) at a time is a long and difficult task. Piaget and Inhelder discuss some research on this process of building three dimensional imagery in Mental Imagery in the Child and in chapter X of the Space book, but much more remains to be done.

Mathematical aspects of the projections also need to be examined. In the objects seen by the doll task, the attention of the child is focused on the point of projection (the doll's eyes) alone, and the plane of projection (what the doll sees) is left quite vague. In the shadow task, both the point of projection (the light source) and the plane of projection (the screen) are specified for the child. In the geometric section task, only the plane of projection (the cut) is mentioned specifically; the identity of the viewer, the point of projection, is not specified. Studies in which the specificity of the point and of the plane of projection are sytematically varied while other variables are held constant would indicate the relative importance of these aspects.

Cross-task comparisons within the same children would also be valuable. The ages given by Piaget are so rough that one can see general patterns of abilities emerging at about the same time but cannot detect any subtleties. For example, children are able to make a projective straight line (experiment 1) before they can succeed on any of the other tasks, and projections of shadows (experiment 3) and sections of geometrical solids (experiment 5) take longer for total success than the others, for they require a stage IV. Whether this is due to the nature of the tasks or to the greater difficulty of the objects used (i.e., that the objects used had greater "resistances") is not clear from the data Piaget reported. But one cannot tell in much detail the relationships among children's performances on experiments 2, 3, 4, and 5 during stages II and III. And this is the period that is most important to the planning of elementary school curricula. In-depth case studies of a few children in the Soviet style might be particularly appropriate here.

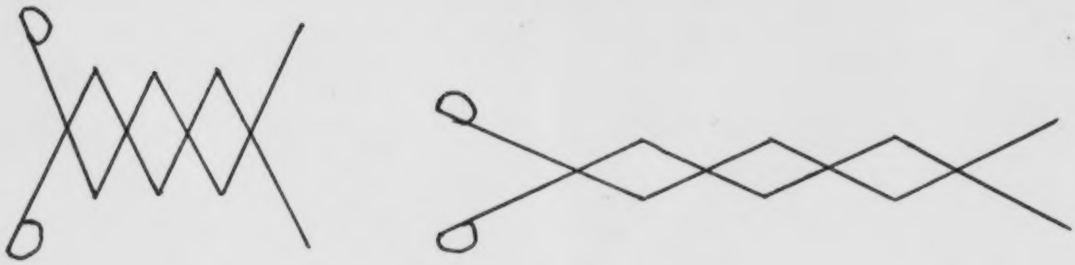
Finally, the first experiment of Piaget, the construction of a straight line, is closest to being a true projective task. The concept of line is a projective concept; however, it is a very complex concept. Intermingled in the idea of line are notions of continuity, collinearity, and rectilinearity. Continuity, the construction of a line segment from adjacent points, is a very difficult one for children. It may be so because a line appears to be an undecomposable whole without any idea of a point involved. This line may be seen to be able to be broken up, but always into smaller bits of a line--small line segments. This idea of an undecomposable whole would be one counterpart of the line most often seen in the world--an edge of some object. Another aspect of the line--rectilinearity/curvilinearity (and planar/not planar) also stems partly from experiences with real objects: things are flat or curved, have a sharp place (an edge) or are smooth, can sit without rocking or cannot. A third source of complexity in the concept of line is motion--objects often move in straight paths. Visual observations of the motions of other people and of objects and kinaesthetic data about one's own motion all can potentially clarify or obfuscate the developing notion of a line.

Future research needs to differentiate clearly among these various aspects of the line. Research which would do this would contribute considerably to our understanding of this important projective concept.

Affine Concepts

Piaget did only one experiment to examine an affine concept--conservation of parallel lines. To do this he studied the child's reactions to a series of affine transformations of linearly connected rhombuses: children were asked to predict the shape of a tool called "Lazy Tongs" as it was opened. This tool is pictured in figure 1. Younger children were given a selection of different sized rods and were asked to construct representations of the tongs ("Make the little windows.") as they would appear as they were opened more. Older children were asked to draw what would happen. Some children were also given a collection of rhombuses and non-rhombuses with the corners covered (so that the sorting process would depend on the parallelism of the sides and not the equality of the angles).

Figure 1. Rhombuses in "Lazy Tongs" closed and open



Stage I (≤ 4)	a child is unable to anticipate any kind of transformation of windows
Stage IIA (4 - 6)	can anticipate change only after seeing the beginning of the opening of the tool; then visualizes an endless enlargement of the "windows"; drawings are often crosses representing the rhombuses;
Stage IIB (6 - 7)	from seeing any position of the tool, can anticipate changes and sees that the windows will get larger and then smaller again; however, the lengths of the sides of the rhombuses change as the shape changes, and the opposite sides do not remain parallel; the transformation is still not a continuous series of changes with changes in height and width inverse to each other
Stage IIIA (7 - 9)	the length of the sides of the rhombuses remains invariant and the sides are parallel, but all the stages of the transformation are not clear;
Stage IIIB (9 - 11)	each stage of the transformation is clearly spelled out
Stage IV (≥ 11)	the whole process of transformation can be deduced a priori, including the point at which the height becomes less than the width

This particular task was an ill-chosen one for several reasons. First, it confounds physical knowledge (having seen the tongs work before) and logical-mathematical knowledge (conservation of parallelism). Second, it involves a transformation which preserves parallelism, but also preserves the length of the sides of the rhombus, and therefore mixes affine and Euclidean properties. A better affine task would preserve only parallelism. The task also seems to involve causality--concern with why the tool works as it does might be distracting. Finally, the task seems to require that the child picture the intermediary steps in a transformation. This is quite difficult, as Piaget and Inhelder report in Mental Imagery in the Child.

Sorting true and false rhombuses with their corners covered, which Piaget described but for which he did not report results, would seem to be a better affine task. Alternatively, a sorting task using rhombuses, trapezoids, and irregular quadrilaterals with other features carefully varied (length of sides, area, etc.) might provide interesting results. Such sorting would, of course, be a perceptual task. However, drawing might be used to examine these shapes on the conceptual level.

In addition to tasks aimed at further investigating the concept of parallelism, table 1 could be used to suggest productive research tasks involving other affine concepts--e.g., convexity, betweenness, conservation of midpoints, ratios of distances along parallel lines. In fact, Martin's study in this volume examines the last listed affine concept--ratios of distances along parallel lines. In particular, it would be interesting to have more information about the parallel development of all the concepts within individual children.

Similarity Concepts

Piaget's work on the development of similarity relations was fairly thorough. It involved five experiments.

Experiment 1: Drawing Similar Triangles Using Parallelism and Ratio

The tasks for this experiment are drawn in figure 2. Solid lines represent what the child is given, and dotted lines show what he is asked to do. Numbers in parentheses show the size of other triangles presented in each task.

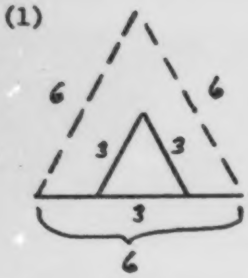
Stage 1 (≤ 4.5)	impossible to carry out useful experiments
Stage IIA (4.5 - 6.5)	drawings take account of neither parallelism nor the equality of the angles
Stage IIB (6.5 - 7.5)	an intuitive idea of parallelism emerges for some particular triangles (equilateral and obtuse isosceles) and there is marked progress in judging the slope of sides of the triangles
Stage IIIA (7.5 - 9)	parallelism of the sides of inscribed triangles is used to construct similar triangles--children move the ruler very carefully to maintain the same slope
Stage IIIB (9 - 11)	begins to use simple proportions (1:2) and to measure the sides of the triangle instead of using parallelism; still uses addition instead of multiplication for difficult ratios (adds equal lengths to all sides)
Stage IV (≥ 11)	can use proportions to construct similar triangles; in particular can do task 4 for more difficult proportions

Experiment 2: Sorting Similar Cardboard Triangles

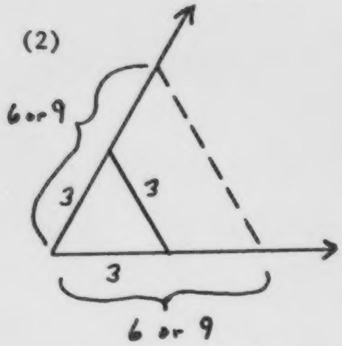
Children were asked to sort cardboard triangles into similar and dissimilar groups. Various sizes and shapes were used. Children could handle these triangles freely and superimpose them to check equality of angles if they wished.

Stage I (≤ 4.5)	impossible to carry out useful experiments
Stage IIA (4.5 - 6.5)	judgments are made globally; sometimes equilateral triangles are put together; even when superimposing

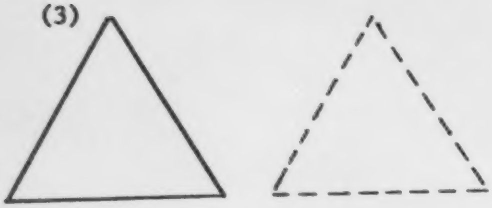
Figure 2. Tasks for experiment 1



Also (3, 1.7, 1.7*)
(3, 6, 6), (6.5, 4, 2.5)

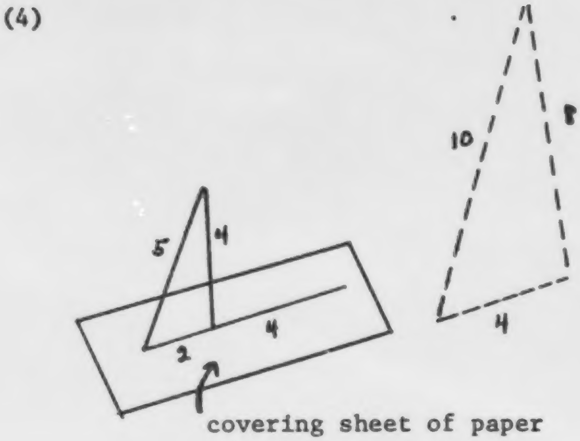


Also (3, 6, 6)*, (4, 2.5, 2.5),
(6, 4.5, 2.5)



Also (3, 6, 6)*, (4, 2.5, 2.5),
(6, 4.5, 2.5)

(5) selection from drawings of
similar and dissimilar triangles



covering sheet of paper

* (base, side, side): other sizes of triangles given in that task
All numbers are in centimeters.

Stage IIA (cont.)	is demonstrated children still focus on the length of sides as the sorting criterion
Stage IIB (6.5 - 7.5)	begins to notice the inclinations of the sides and even sometimes characteristics of the apex angle (e.g., refer to the thinness of a triangle); when shown how to superimpose triangles, sometimes look at the apex angle but not used very accurately
Stage IIIA (7.5 - 9)	spontaneous superimposing; classification by parallelism of sides is successful; a beginning of focusing

Stage IIIA (cont.)	attention on angles
Stage IIIB (9 - 11)	classification by superimposing the angles in a triangle

Piaget remarks that he obtained similar stages when he had children sort similar cardboard rhombuses, but he does not report any details of this experiment.

Experiment 3: Choosing and Drawing Similar Rectangles

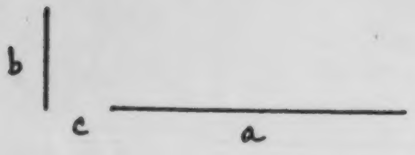
Two kinds of tasks were involved. In the first, children were shown a standard drawing of a rectangle 1.5 cm by 3 cm and comparison drawings of rectangles 4 cm in width but with lengths varying from 6 to 15 cm (8 being correct). They were asked to choose which drawing was "the same shape but bigger" or the "daddy of the little one." A magnifying glass was sometimes used to demonstrate enlargement. In the other task, the child was asked to draw on another sheet of paper a box of the same shape but larger than the standard drawing. Sometimes a baseline of double or triple length was given. To clarify the request for a larger similar figure, children often practiced by classifying similar and non-similar rhombuses.

Stage I (≤ 4.5)	outside the possibilities of experimentation
Stage II (4.5 - 7.5)	selected and spontaneous drawings: both have the ratio of length to height greater than it is--almost as if a child thinks, "The more rectangular (i.e., longer) it is, the better."
Stage IIIA (7.5 - 9.5)	perceptual comparisons become fairly accurate, but drawings are not; centrations alternate between length and width; spontaneous attempts at measurement
Stage IIIB (9.5 - 11)	uses measurement in drawings; adds equal amounts to the length and width by drawing and then alters the drawing to match his perceptual estimate; drawings correct for simple ratios (1:2)
Stage IV (≥ 11)	uses multiplication to make correct drawings involving more difficult proportions

Experiment 4: Drawing a Similar Configuration of Line Segments

Piaget tried one other experiment involving proportionality in which he tried to eliminate the role of the strong configuration in the triangle and rectangle experiments. A drawing like that in figure 3 was presented, and children were asked to select correct enlargements and to draw them.

Figure 3

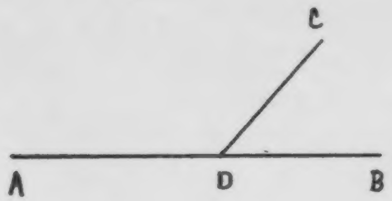


The stages matched those in the rectangle experiment. By Stage IIIB, a and b were drawn correctly, and by Stage IV, c was also correct.

Experiment 5: Copying Supplementary Angles

One experiment reported by Piaget in The Child's Conception of Geometry is also relevant here. Children were asked to reproduce on another sheet the drawing in figure 4. They could study the figure as often as they wished except when they were actually drawing (the model was placed behind the subject). Thus, direct visual comparison was eliminated. Rulers, strips of paper, string, cardboard triangles (for angles), and compasses were provided to be used in measuring.

Figure 4



Stage I (≤ 4.5)	drawings made by visual estimate
Stage IIA (4.5 - 6)	visual estimate, no attempt at measurement even when it is suggested
Stage IIB (6 - 7)	AB or CD or both were measured, but not AD or DB or the angle
Stage IIIA (7 - 8.5)	AD and DB are measured, and children tried to maintain the slope of their ruler when drawing CD
Stage IIIB (8.5 - 11)	AC and CB were measured to fix point C
Stage IV (≥ 11)	most children dropped a perpendicular from C and used it to find the location of point C

Summary of Similarity Concepts

This set of experiments examines the basic set of ideas in similarities--angles and ratios of lengths along non-parallel lines. The tasks also involved shapes whose equal angles will determine similarity (triangles) and those that will not be so determined (rectangles). The tasks involving picture selection were perceptual, and the drawing tasks were conceptual. Tasks which fall between these two (into representation) might involve the construction of similar figures using sets of equal sticks (the ratio would be determined by the number of sticks used per side) or using sets of unequal sticks (the ratio would then be determined by the size of the sticks used). Alternatively, figures might be constructed on identical or similar geoboards. Such research studies would relate directly to the schools, for such tasks would be good geometric activities for children.

Piaget's similarity experiments seem more relevant to school mathematics than the projective or affine experiments. This is because similarity ideas are included in many parts of the school curriculum. Some models for rational number concepts are based on similarity; thus, part of students' difficulty with rationals may stem from problems with similarity ideas. Ratio and proportion are part of the school curriculum from at least the seventh grade on, and they present many difficulties to the student. Standardized tests include many proportion word problems. Verbal analogies (a:b::c:d) form major parts of many intelligence tests. Similar geometric shapes would seem to provide a helpful mental image for other types of proportion and analogy situations. Training studies of teaching experiments concerning ways to teach geometric similarities and ways to generalize the solution of geometric proportions to other types of proportion would be valuable.

Experiment 2 is the only one of Piaget's similarity experiments that directly addresses the topological to Euclidean assertion. In Stage IIIA children classify similar triangles by the parallelism of their sides, and not until later (Stage IIIB) do they classify by comparing the angles. Here the affine concept, parallelism, is used before the similarity idea, equality of angles. Thus Piaget's evidence for the progression of the middle geometries from projective to affine to similarity primarily consists of two experiments: the first projective experiment which demonstrates that the projective straight line can be constructed by age 7 and the second similarity experiment which indicates that parallelism is used in classifying triangles by age 7.5 to 9 and angles are used by ages 9-11. This evidence is intriguing, but certainly not sufficient. Other studies which examine different projective, affine, and similarity concepts are needed, especially those which examine all three types of concepts in the same children.

Conclusion

Piaget performed a valuable service by pointing out the importance of different viewpoints and of children's conscious awareness of different viewpoints. He has offered an intriguing hypothesis in the topological-Euclidean ordering, one that seems to provide a productive array of research questions. His experiments always offer great insight into the ways children think and act. But people with mathematical training need to design and carry out studies which will elucidate and begin to answer the questions Piaget has raised concerning projective, affine, and similarity concepts.

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The Child's Concept of Ratio of Distances

--an Affine Invariant

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Felix Klein's Erlanger Programm has been used to classify geometries according to their subgroup relationships. Accordingly, the affine group is a subgroup of the projective group and contains the group of similarities as a subgroup (see figure 1). The Programm has also been appealed to recently by mathematics educators and psychologists as modeling the sequence of the child's construction of space. Piaget maintains that the child constructs his conceptual space in an order roughly paralleling the classificiation of geometries. That is, to Piaget, the child's first spatial constructions are topological. Projective and Euclidean concepts develop later.

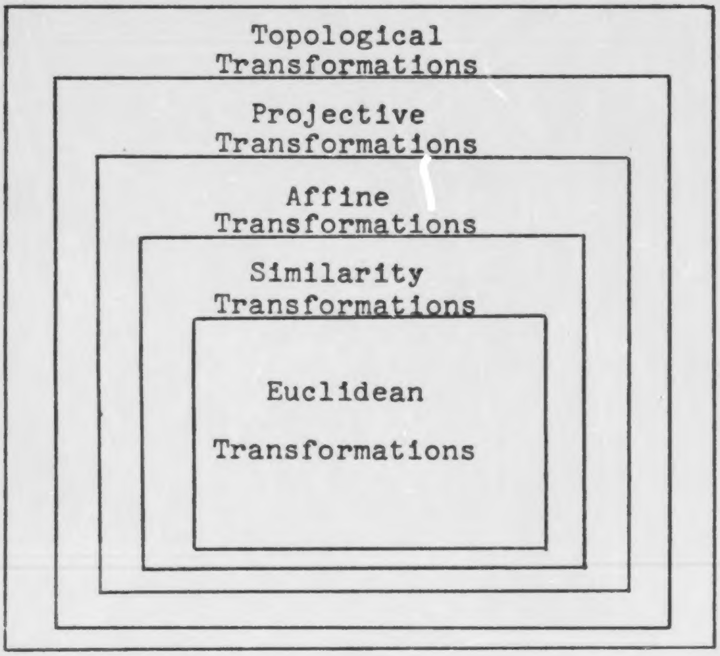
Topological Transformations
Projective Transformations
Affine Transformations
Similarity Transformations
Euclidean Transformations

Figure 1. A hierarchy of transformation subgroups.

There have been a few replications of Piaget's work investigating the child's concept of space. There have been fewer tests of his theory. Mathematics educators must examine the structure of the mathematics involved in the various geometries and ask questions about the consequences of the developmental sequences suggested by Piaget. Applying this sequence to the mathematical structure can lead to many research questions (Martin, 1976).

Since similarities form a subgroup of the affine group, if Piaget's suggested sequence is essentially correct, one would expect affine concepts to develop prior to similarity concepts. Parallelism is associated with affine geometry. Angle measure and proportionality are associated with similarities. However, affine transformations also preserve ratios of distances, provided the distances are taken along the same line or along parallel lines.* Similarity transformations conserve ratios of distances in all directions. Two natural questions arise: (1) When do children develop the ability to conserve ratios of distances in one direction?, and (2) Does this ability develop prior to the ability to conserve ratios in all directions? This study deals primarily with the first question. Attempts to answer the second question will be made by comparing the results of this investigation to the results from Piaget's similarity experiments (Piaget, 1967).

Related Literature and Mathematical Considerations

Fuson (this volume) reviews some of Piaget's (1967) research dealing with affine and similarity concepts and notes the paucity of related studies. Piaget's lone affine task is the now familiar "Lazy Tongs" task. As Fuson notes, the transformation involved conserves length as well as parallelism and thus contains Euclidean elements. Piaget recognizes that he is utilizing a particular type of affine transformation and reports the children's development in terms of both parallelism and the invariant length of the sides. Though Piaget is aware that length is not an affine invariant (p. 305), he evidently does not view the fact as a major obstacle to a study of the child's representation of parallel lines and of his concept of parallelism. On the basis of the results for this task, he reports that the concept of parallelism is acquired at about 7 or 8 years of age. Since Piaget defines affinities as "projective correspondences conserving parallelisms" (p. 301), he then turns his attention to similarities ("affinities conserving angles" (p. 301)). More will be said about his similarity tasks later.

Although parallelism is an affine invariant, this investigation deals with the affine invariant of ratios of distances in one direction, that is, ratios of distances on the same line or along parallel lines. Piaget does not address himself to the study of this concept.

A discussion of ratio and proportion is appropriate here. "Ratio" is defined as the quotient of two numbers. More generally, the term conveys the notion of relative magnitude. The conservation of a ratio of distances is illustrated in figure 2. Since affine transformations preserve ratios of distances on the same line or parallel lines, they, in particular, send

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* For a detailed list of invariant properties associated with transformation groups, see Fuson, this volume.

equal distances into equal distances, thus preserving midpoints. The rectangle ABCD is affinely equivalent to the parallelogram A'B'C'D'. The ratio AP/PB is equal to A'P'/P'B'. These are ratios of segments along the same line and its image, respectively. The ratio AP/AR is not equal to the ratio A'P'/A'R'. These are ratios taken neither from the same line nor along parallel lines. A similarity transformation would have preserved ratios in all directions.

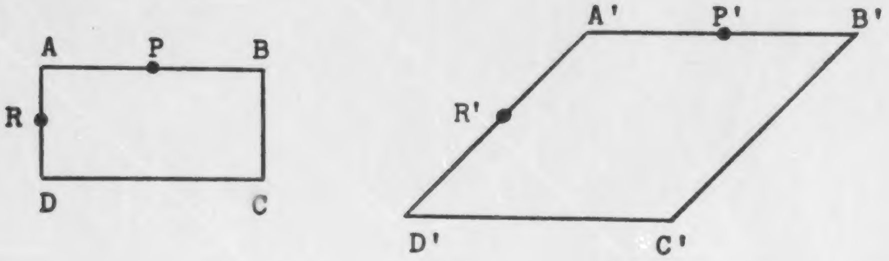

Figure 2. Ratios of distances preserved

A "proportion" is an equivalence of two ratios. Many mathematical models used to describe physical reality involve proportionality. Studies on proportion (Lunzer & Pumfrey, 1966; Lovell & Butterworth, 1966) have concluded that the schema of proportionality is of a higher order than the schema of ratio. This is reasonable to expect and is consistant with Piagetian theory, proportion being a relation between relations.

Consider the proportion $a/b = c/d$. To focus on the geometric aspects of ratio and proportion and place the discussion within the context of this investigation, suppose rod R_1 is divided into "b" congruent segments. The ratio a/b could be used to denote the relative magnitude of "a" segments to the total length. How could the situation be varied to obtain the proportion $a/b = c/d$? Rod R_2 could be congruent or identical to R_1 but subdivided into "d" ($\neq b$) segments. Then $a/b = c/d$ if but only if $a \cdot d = b \cdot c$. In this case $b \neq d$ and $a \neq c$ (e.g., $2/3 = 4/6$), but the segment of R_2 which is "c" segments long is congruent to the segment of R_1 which is "a" segments long.

Another possibility would be to choose rod R_2 of different length than R_1 but divide it into "d" ($=b$) congruent segments. The ratio c/d would represent the relative magnitude of "c" segments to the total length. Then $a/b = c/d$ because $a=c$ and $b=d$. But the segment of R_1 which is "a" segments long would not be congruent to the segment of R_2 which is "a" segments long.

These two proportions should be quite different psychologically.

Note that in the former case, the unit remained the same but it was divided into a different number of segments. This type of proportion usually appears in the middle school curriculum. In the latter case the size of the unit changed but the number of subdivisions remained the same. It seems appropriate to designate this second type of proportion *conservation of a ratio*.[*] This type of proportion is less frequently encountered in textbooks. Though unsuitable in a context of generating equivalent fractions, it is highly relevant to geometry. For example, if a polygon P_1 is similar to a polygon P_2, then the ratio of the lengths of two of the sides of P_1 is conserved in the ratio of their respective images. The ratios are identical.

In a series of investigations, R. Karplus et al. (Karplus, R. & Peterson, 1970; Karplus, R., & Karplus, E. F., 1972; Karplus, E. F., Karplus, R., & Wollman, 1974; Wollman & Karplus, R., 1974) studied the development of the child's concept of proportionality. Although they used several different tasks, only one will be described here as illustrative. Subjects were given a picture of a stick figure, called Mr. Short, and some paper clips. They were told of another stick figure, Mr. Tall, similar to Mr. Short but taller. The experimenter stated, "I measured Mr. Short's height with large buttons, one on top of the other, starting with the floor between Mr. Short's feet and going to the top of his head. Four buttons reached to the top of his head. ...Then I measured Mr. Tall with the same buttons and found that he was six buttons high." The subjects' task was to measure Mr. Short's height in paper clips and predict the height of Mr. Tall in paper clips and then explain how they figured their prediction. Although this particular task was group administered and used paper and pencil, other tasks were given during individual interviews.

Mr. Short's height measured in paper clips was 6. This could give rise to the proportion:

$$\frac{4 \text{ buttons for Mr. Short}}{6 \text{ buttons for Mr. Tall}} = \frac{6 \text{ paper clips for Mr. Short}}{x \text{ paper clips for Mr. Tall}}$$

How can this proportion be interpreted? Is the unit subdivided into a different number of segments on each side of the equality? Is the unit different on each side? What is the unit? One possible interpretation is that the unit is Mr. Tall or a line segment of length equal to the height of Mr. Tall. This unit is divided into six segments. Mr. Short could be thought of as a subset of the unit consisting of four of those segments. The unit is then subdivided into small segments, each of length equal to the length of a paper clip. An alternate choice of the unit could be Mr. Short. There are other possible choices for the unit. Indeed there are other possible proportions for representing the problem. This discussion merely serves to demonstrate some of the possible complications involved in proportional reasoning.

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[*] Some would not label this second type a proportion because the unit has changed.

Lunzer and Pumfrey (1966) and Lovell and Butterworth (1966) have found that proportional reasoning unaccompanied by physical actions are rarely used before age 15. Karplus, in the studies cited earlier, found only 10-15% of seventh and eighth graders consistently used proportional reasoning to solve his tasks. He suggests that factors such as the ratio involved and the "cognitive style" of the subject also contribute to the strategy used. The Karplus studies are recommended both for their longitudinal nature and for their interesting analyses of proportional reasoning.

Piaget (1967) maintains that proportionality is more easily studied in spatial problems than in nongeometrical situations. Here the origins of proportional reasoning can be found in the actual perception of figures. Once again, however, the distinction must be made between perception and representation. "Assuming that the child can perceive two models as similar, will he be able to draw them in proportion, not only as regards the overall appearance, but preserving the angles and relative dimensions?" (p. 322) Since Fuson has already summarized Piaget's similarity experiments, only a few points of particular relevance to this investigation will be highlighted. First, strategies utilizing parallelism are used prior to strategies based on congruent angles for solving geometric proportion problems. The parallelism strategy surfaces around 8 or 9 years of age with the angle strategy following between the ages of 9 and 11. Second, in the case of selecting or producing similar rectangles, perception is fairly accurate at age 7.5 - 9.5 years but drawings are not. The drawings are rectangles, however, and thus conserve angles, parallelism, and ratios of distance along parallel lines. Early attempts at proportional reasoning at around 9.5 - 11 years of age take the form of adding equal amounts to the length and to the width. Drawings are more likely to be correct for simple ratios such as 2:1. (Karplus also reports a widespread use of additive reasoning among his subjects and a higher rate of success for the simpler ratios.) Piaget's subjects attained a stable quantitative concept of proportionality at around 12 years of age. Piaget thus reports three phases in the development of similarities and proportions: (1) parallelism of sides, (2) equality of angles, and (3) metric proportionality.

Procedures

The Sample: The sample consisted of 40 subjects, 20 from grade 5, and 10 from each of grades 3 and 4. The children were randomly selected from four elementary schools chosen by the Associate Superintendent of the Joplin, Missouri, public school district as representative of the district. The schools were "representative" in a socio-economic sense, their composite student population forming a heterogeneous group ranging from upper-middle to lower socio-economic levels. Five children were selected from each school at each grade level. Originally it was planned to use 20 subjects from each of grades 3, 4, and 5. However, the uniformly poor performance of the third and fourth graders on the tasks made it unnecessary, in the opinion of this investigator, to interview the remaining 20 children.

The average age of the fifth graders was 11 years, 1 month; the average age of the fourth graders was 9 years, 8 months; and the average age of the third graders was 8 years, 11 months. The standard deviation was 4 months for grades 4 and 5, and 6 months for grade 3. Based upon Piaget's findings, these ages were selected as most likely to yield measurable results on affine tasks, or more particularly, on conservation of ratios of distances.

Table 1

Conservation of Ratio of Distances Test

Item	Model	Copies
1	35 cm, bead at 25	28 cm, bead at 15 bead at 18 bead at 20 bead at 22 bead at 25
2	21 cm, bead at 15	same as on item 1
3	28 cm, bead at 20	21 cm, bead at 15 35 cm, bead at 27 42 cm, bead at 28
4	28 cm, bead at 20	21 cm, bead at 13 35 cm, bead at 25 42 cm, bead at 33
5	28 cm, bead at 20	21 cm, bead at 13 35 cm, bead at 27 42 cm, bead at 30
6	35 cm, bead at 25	place bead on 42 cm rod
7	21 cm, bead at 15	place bead on 14 cm rod
8	38 cm, bead at 20	place bead on 35 cm rod
9	28 cm, bead at 20	place bead on 21 cm rod
10	28 cm, bead at 20	draw rod same size
11	28 cm, bead at 20	draw a shorter rod
12	21 cm, bead at 15	draw a longer rod

The Test: The test was composed of 12 items (see table 1). The first five items consist of a model and copies. In each case the model is a rod 3mm in diameter with a bead placed on it dividing the rod into two segments having lengths in the ratio of 5:2. In other words, the bead is 5/7 of the way from the end of the rod to the other. The copies are also of 3mm diameter and have beads located as in table 1. The task is to select the copy which preserves the ratio. The copies are constructed so as to yield clues of the strategies used by the subjects. For example, he may conserve the distance from the bead to one end or the other of the model rod on his choice of copies. Or he may rely solely on perception and overestimate or underestimate the distance from the end to the bead on the copy. Item 1 requires selection from copies all shorter than the model; item 2 requires selection from copies all longer than the model; items 3-5 require selection from copies some longer, some shorter than the model.

Items 6-9 require the subject to construct copies of a model by placing a bead appropriately on a rod of different length from that of the model. Items 10-12 require the subject to draw copies the same size as, shorter than, or longer than the model with the bead drawn appropriately.

Administration: During individual interviews with each child the items were presented in order 1-12. The interview began with two warm-up tasks. Since the warm-up tasks were designed to convey the intent of the tasks to follow, the interviewer used a limited intervention format giving suggestions to or asking leading questions of the subject. Such an approach was justified on the basis that a child is not likely to develop a scheme for conserving ratios of distance from such minimal assistance.

Before beginning the warm-up tasks, the investigator told the subject that there were no right or wrong answers to the upcoming questions, and that he (the investigator) just wanted to know how he (the subject) felt about the answers. In the first warm-up task, the subject was presented a model rod 25 cm long. The rod was subdivided into five congruent segments by ink marks on the rod. A bead was on the rod dividing the rod into lengths in the ratio 3:2 (i.e., the bead was on the third mark). The subject was asked to imagine that the rod "gets shorter and shorter until it is just this size." At this point a second rod, shorter (20 cm in length) than the first, was given to the child. It, too, was subdivided into five congruent segments but contained no bead. The child was asked to put the bead on the second rod where he thought it should be. The task was repeated requiring the subject to place the bead on a rod longer (30 cm) than the original. If the child did not respond by placing the bead on the third mark of the longer rod, the investigator called the subject's attention to the marks on the rods. In the discussion which followed, the investigator asked such questions as, "If we cut the sticks at each mark, how many pieces would we get?"; "Would they all be the same length or would they be of different lengths?"; and, "How many pieces are there on that side of the bead?... How many on the other side of the bead?" After the discussion the task was repeated. Effects of this intervention are discussed later in this paper.

In the second warm-up task, the subject was presented a model rod 28 cm long. A bead was located 21 cm from one end. There were no marks on the rod, thus no convenient subdivisions. Instructions similar to those on the first task were then given and the subject was asked to place the bead appropriately on an unmarked rod of length 24 cm. While the subject investigated the model and the copy rods, the investigator commented on the absence of the marks and placed a "helper stick" alongside the model. The "helper stick" was of the same length as the model and was subdivided into four congruent segments, but contained no bead. Placing the aid alongside the model revealed that the bead on the model would be on the third mark if the model was marked, thus dividing the model rod into segments in the ratio of 3:1. The investigator suggested that perhaps the "helper stick" could be useful and presented two more "helper sticks" of length equal to the unmarked copy rod. Both of these latter aids were subdivided into four segments but only one was divided into congruent segments. The subject was then allowed to use the aids. The task was repeated in a similar fashion with the same model rod but with an unmarked rod of length 32 cm upon which to place the bead. Analogously, aids were presented.

Regardless of which "helper stick" the subject used, that is, the one marked in congruent segments or the one in noncongruent segments, the investigator asked questions such as, "Could we use the other helper stick?... What would happen if we were to do so?...Would it make any difference or would the bead go in the same place?" These questions and the ensuing discussion were intended to make the subject aware of the fact that the two aids were different. However, the subject was not told which one to use.

None of the rods used in the actual test items were marked. However, a collection of "helper sticks" and a meter stick were available during the administration of each item. The collection contained 10 rods, two of each of the lengths 14, 21, 28, 35, and 42 cm. Thus for each rod used either as a model or a copy there were two possible "helper sticks." All of the potential aids were subdivided into seven subdivisions. One aid of each length was divided into congruent subdivisions, the other serving as a mislead. The children were told the "helper sticks" were to use if they wanted to do so. The time required for each interview was 20-35 minutes. Increased use of the aids resulted in an increase in the amount of time required.

Results

Of the 10 third graders, none answered more than two of the first nine items correctly. Of the 10 fourth graders, none answered more than three of the first nine items correctly. The variability was much higher for the fifth grade scores. Seven of the 20 fifth graders responded appropriately to all of the first nine items. On the other hand, two of the fifth grade children made scores of zero. Mean scores for the third, fourth, and fifth grades were .9, 1.1, and 5.85 respectively. These data together with a frequency distribution are displayed in tables 2 and 3.

Since no exact length for the drawings was established physically by any proposed copies in items 11 and 12, it was necessary to define what a correct response would be. An error of ±1 cm was allowed for the placement of the bead on a rod of length 28 cm. This rod was selected because of its predominance in the tasks. The error amounts to approximately 4% of the length of the rod. This same percent error was then applied to whatever length the subject drew. That is, in order to be counted as correct, the bead had to be located in the drawing at a point

$$(5/7 \pm .04) \cdot \text{Length}$$

from one end of the rod. (This same criterion was also applied to the location of the bead for items 6 through 9.)

All but two of the third graders were able to draw a copy the same size as the model (item 10). Only one of the fourth graders failed at this task and none of the fifth graders failed. Three third graders succeeded on item 11 (smaller copy) and three on item 12 (larger copy). However, none were successful on both. One fourth grader drew an appropriate copy for item 11 and three drew appropriate copies for item 12. Again, no one succeeded on both tasks. Ten fifth graders conserved ratio of distances in their drawing of a copy shorter than the model and 13 did so in their drawing of a copy longer than the model. Nine drew appropriate copies for both. Four of these latter nine also had made perfect scores on items 1-9, one had missed only the first item, and one had missed only the first two items.

Table 2

Frequency Distribution by Grade for Items 1-9

	Score										Total
	0	1	2	3	4	5	6	7	8	9	
Grade 3	4	3	3	0	0	0	0	0	0	0	10
Grade 4	3	4	2	1	0	0	0	0	0	0	10
Grade 5	2	0	3	0	1	3	1	1	2	7	20
Total	9	7	8	1	1	3	1	1	2	7	40

Table 3

Means and Standard Deviations by Grade for Items 1-9

	Grade 3	Grade 4	Grade 5	Total
Mean	.9	1.1	5.85	3.42
Std. Dev.	.83	.94	3.15	3.35

A measure other than score of each child's conception of conservation of ratio is the strategy employed in responding to the tasks. Only three third graders made any use whatever of the available aids. Five fourth graders used them on at least one item, but three of these on only one item and none on more than five. In contrast, 19 of 20 fifth graders employed the aids and 18 of these used them in some fashion on every item. More will be said about the various uses of the aids in the next section.

The most frequently observed incorrect strategy was the conservation of the distance from one end of the rod to the bead. Eight subjects conserved the distance from the bead to the right end of the rod on their copies on at least five of the first nine items. Four of these eight were third graders, three were fourth graders, and one was a fifth grader. Conservation of the distance from the left end to the bead was less frequent and never occurred as a consistent strategy. It was used most frequently by one fourth grader who applied it to four items in combination with conservation of the distance from the right end to the bead on two more items. On the drawing items one fifth grader conserved the distance from the bead to the left end of the rod on item 11 and one conserved the distance to the right end on item 12. None of the third or fourth graders did either on their drawings.

Conclusions

Scores and strategies of the fifth graders were dramatically different from scores and strategies of the third and fourth graders. The fifth graders used the aids. The younger subjects did not. Eleven of the fifth graders had a success rate of 75% or higher. None of the third or fourth graders had a success rate as high as 50%. One of the major questions of the study was, "When do children develop the ability to conserve ratios of distances in one direction?" To the extent that the test used in this study measures this ability, the answer appears rather clearly to be around age 11.

To answer the second major question, "Does this ability develop prior to the ability to conserve ratios of distances in all directions?", it is necessary to re-examine Piaget's similarity experiment results. Piaget found some quantitative idea of proportions present among children younger than 11 years of age but only for simple tasks and simple ratios such as 1:2. But it was not until Stage IV, about 11 years of age, that his subjects applied quantitative proportional reasoning consistently. Thus, seemingly Piaget's subjects succeeded on his similarity tasks at about the same age as did the subjects in this investigation.

To provide better answers to the second question, further investigation is desirable. One suggestion is to use two sets of tasks, one requiring conservation of ratios of distances in one direction as in this investigation, followed by a set of similarity tasks requiring conservation in all directions. Probably the sample should use fifth and sixth graders. Such a study could attempt to identify children who can conserve ratios of distances in one direction (affine invariance) but who cannot conserve ratios of distances in all directions simultaneously (similarity invariance).

A comment should be made about the effect of the warm-up tasks. Most of the subjects, even the younger ones, were able to use the aids appropriately during the warm-up tasks after discussion. Also the younger subjects were often able to justify using the "helper stick" with the congruent segments as better than using the "helper stick" with noncongruent segments. Yet, only the fifth graders were able to use the aids on the actual test. One interpretation of the fifth graders' success is that they learned a strategy during the warm-up tasks. But children of all ages used the aids in the warm-up tasks. Why did none of the third or fourth graders learn the strategy?

An alternate interpretation is that the fifth graders recognized in the aids a means to apply mental operations available to them which were not yet available to the third and fourth graders. This position is supported by the fact that during the warm-up tasks it was not necessary to find an appropriate aid for the model. It was provided by the investigator, and it was necessary to select from only two "helper sticks" the appropriate one for use in making the copy. During the actual test all of the aid rods were together and it was necessary to select one appropriate for use with the model and one appropriate for the copy. On items 3 - 5 this meant comparing aids to copies of three different lengths so the subject could not even know which length aid he needed. Therefore, in the opinion of this investigator, it is this latter interpretation which should be made.

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Cognitive Studies Using Euclidean Transformations*

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When a mathematics educator tries to determine a sequence of appropriate instructional experiences leading to a mathematical concept, he often performs a Gagné-type analysis of the concept. That is, the educator creates a hierarchy of "learnings" which must be mastered before terminal concept attainment is possible. Invariably prerequisite learnings are included in the hierarchy if they are logically (mathematically) more primitive than the desired learning and if these mathematically determined prerequisites generally fall within the scope of the curriculum at a given level.

This method of modeling instructional structure almost solely on mathematical structure has been used for years and has met with a certain amount of difficulty. Perhaps a rethinking of certain assumptions underlying this approach is in order. Specifically, a large part of the difficulty may lie in the tacit acceptance of two unwarranted assumptions: first, that people structure concepts in a manner that is mathematically logical; and second, that children structure concepts in the same way that adults do.

With regard to the first assumption, Lesh (1976) points out that there are some striking dissimilarities between general mathematical structures and children's basic cognitive structures. He maintains that the structures children use when first mastering a concept are "...'messy' structures that do not give rise to neat tidy theories...It seems likely that mathematical descriptions of many children's concepts will involve structures that mathematicians have not bothered to formalize" (p. 50). Indeed, Weinzweig's contribution to this volume is a careful development of a mathematical model of the cognitive structures used when children deal with isometries, and the framework of his model is considerably more involved than the usual mathematical structure for the group of rigid motions.

Dienes has consistently argued that children think differently than adults:

> Adults tend to think analytically, which means that they tend to see a structure as a set of relationships and to understand it by analyzing it into these component relationships. Children tend to think constructively, which means that they tend to build up their structures from separate components. All the while they have an intuitive idea or 'feel' for the kind of end-result they desire. This disparity between child and adult often results in a lack of effective communication between teacher and child. It is as though they were at opposite ends of a long tunnel or that each were looking at opposite sides of the same coin. (1971, p. 58)

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If, in fact, the above views are valid, then present methods of curriculum design should be revised. The Gagné approach, in essence, reduces a mathematical concept to the sum of its logical prerequisites. Perhaps this position is tenable in the mathematical realm, where, for example, the juxtaposition of $p \rightarrow q$ and $q \rightarrow r$ inexorably leads to $p \rightarrow r$. However, when human beings enter into an educational process there is yet another important factor that must be taken into consideration when developing instructional structures, viz. the cognitive structure of the learner, be he adult or child.

To further explore the above positions, three studies were designed to investigate the nature of specific mathematical concepts as they relate to the cognitive structure of the learner. The first study, by Moyer, uses isometries to demonstrate that the spontaneously developed cognitive structures that young children bring to bear when performing Euclidean transformations are not totally compatible with the mathematical structure of the group of rigid motions. The second study, by Moyer and Johnson, analyzes the performance of older subjects on a three-dimensional task in transformation geometry. Results indicate that the difficulty of the task is not predictable on the basis of mathematical structure alone--even though the subjects are somewhat older and might be expected to possess structures more in accord with "typical" mathematical structures. Finally, the third study, by Johnson, gives an example of an instructional structure that can be used to bridge the gap when the mathematical structure of the task and the cognitive structure of the learner are not entirely compatible.

The First Study: Two-Dimensional Isometries and Young Children

Piaget (1971) has indicated that the spontaneous intuitive structures that young children automatically build through interaction with their environment correspond quite closely with structures that mathematicians have developed. The problem is to determine which mathematical structures have cognitive correlates at what age levels, and to what extent. In this first study, Moyer (1974) set out to determine the extent to which the structures that children utilize when performing Euclidean transformations correspond to the mathematical structure of isometries. Specifically, three areas of possible correspondence were explored. These are reflected in the three major questions researched.

(1) "Is a child's understanding of isometries dependent upon an explicit awareness of the physical motion related to a given transformation?" Mathematically, motion is not involved. Euclidean transformations are 1-1 correspondences between (static) images and their (static) pre-images. Cognitively, however, for children (or even adults) it may be that the essential nature of Euclidean transformations involves a visualization of the actual motion involved.

(2) "Which invariant aspects of a given Euclidean transformation are perceived as most critical?" Mathematically speaking, topological invariants are more primitive than Euclidean or projective invariants. Hence, mathematical structure would predict that a child unable to conserve the Euclidean aspects of a given transformation might nonetheless feel very

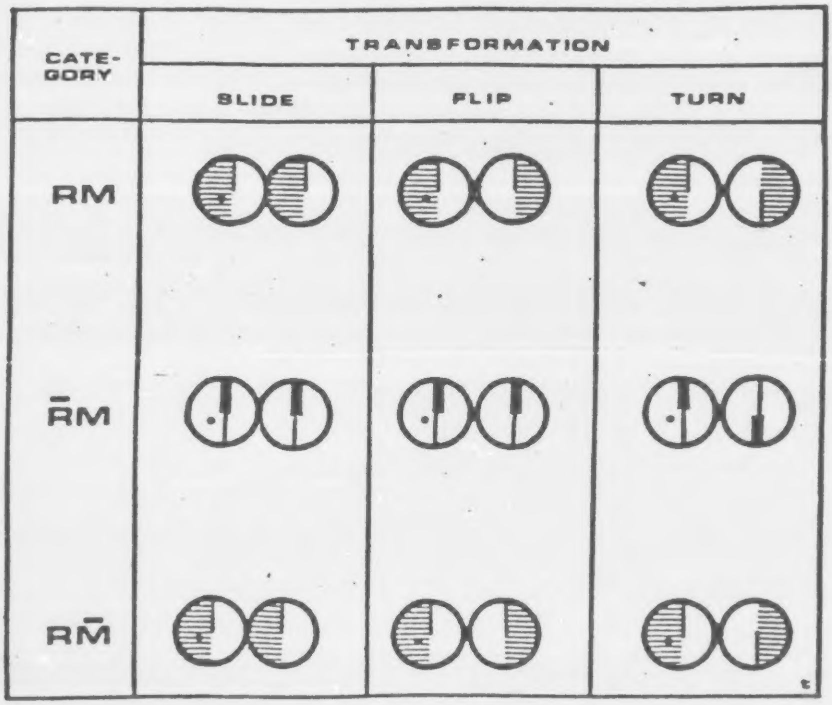
comfortable in conserving the topological aspects.

(3) "Are Euclidean reflections easier for children to deal with than translations and rotations?" According to mathematical structure, reflections are mathematically more primitive than rotations or translations because compositions of appropriate reflections give rise to the other two types of transformations. Similarly, rotations are more primitive than translations. But, these facts do not necessarily imply that the order of difficulty of children's transformation tasks are: reflections (easiest), rotations (next easiest), translations (hardest). In fact, the opposite order of difficulty could perhaps be a more accurate description of children's performance on transformation tasks.

Procedure

The study involved 120 preschool, kindergarten, first, second, and third grade children. Two identical, specially marked plastic disks (12 inches in diameter) were set before each child. The experimenter placed a dot on the left disk. The child was then asked to place a dot on the right disk so that the two disks would remain identical. The child was given nine tasks, for which the disks were variously placed so that they could be shown congruent by a slide, a flip, or a turn. Further, in some instances the child was shown the motion (M) necessary for the superposition of the circles; in others he was not ($\bar{M}$). Finally, some of the trials utilized disks which were shaded half-red (R); others used disks without any red ($\bar{R}$) (see figure 1).

Figure 1. Disposition of the circles for each of the nine tasks immediately prior to subject's response



Results

The results showed that cognitive ability is in accord with mathematical structure in the first two areas investigated. First, explicit awareness of the motion associated with a transformation neither helped nor hindered a child's ability to perform a task. This is in accord with the mathematical structure in that mathematics is also oblivious to the actual motion involved. This interesting result is limited to the tasks used in the study. However, it does point the way for further research with different transformations using different materials. The result of such testing could have strong implications for the current emphasis on the motion aspect of the Euclidean transformations done in the elementary school. For example, if further research indicated that the motion involved in Euclidean transformations is a confusing element, then instruction might be better developed from a 1-1 correspondence point of view.

Second, the topological property of enclosure (as illustrated by the dot in the red portion) was significantly important for children regardless of age, while the use of the projective and Euclidean cues (e.g., the distance of the dot from the center) increased with age. This, too, is in accord with mathematical structure since topological invariants are more primitive than Euclidean or projective invariants. The fact that this research indicates that children "notice" the topological features of the transformations developmentally prior to projective or Euclidean features is a further indication of the correspondence that exists between mathematical and cognitive structures involved.

However, results also indicated that cognitive development departs from that which would be predicted by mathematical structures. This was shown with regard to the developmental order in which isometries are mastered by the children: slide first, then flip, and finally, turn. Since the research showed that the children's ability to perform the slide tasks is the first to develop, cognitively speaking it must be concluded that the flip is not primitive.

Conclusions

These results clearly point to areas of consonance and dissonance between the structures children bring to bear on isometric tasks and the structure of the tasks as set down by mathematicians. It must be concluded that children come to understand certain concepts (isometries in this case) in ways that are distinct from the ways in which a logical development might proceed. Hence, to base an entire curriculum strictly on logical development may not be the most economical way to proceed. In the case at hand it was shown that the reflection does not form a basis for the other transformation types. To build a curriculum of isometries with the flip as the basic element is probably not the most auspicious method. In fact, children may not conceive of isometries as being slides, flips, or turns. That is, children may not think of isometries as being rigid motions but may instead focus on changes in relationships within the (static) initial configuration and the (static) final configuration. But what are these primitive relations that children use? This study indicates that the primitive relations may not be the typical Euclidean relations (involving fixed points of reference, distance relations, and straight lines) that adults tend to favor but may instead correspond loosely to more general mathematical relations--like topological relations such as enclosure (or "onto-ness").

It will be difficult to devise maximally effective instructional activities concerning transformation geometry concepts until research can disclose what children consider to be the essential characteristics of rigid transformations. In the present monograph, the studies by Schultz, Thomas, and Kidder were aimed at providing more information about these primitive relations that children use.

The Second Study: Three Dimensional Transformations with Older Subjects

A second study (Moyer & Johnson) investigated the same general issue of whether it is possible to predict the difficulty of a mathematical task by considering only its logical structure. This study investigated the performance of subjects beyond the fourth grade level on a three-dimensional transformation task. Older subjects were desirable because of the possibility that the way they organize mathematical concepts might be closer to mathematical structures than was observed in younger children. One reason a three-dimensional transformation task was chosen was that little research has been done by mathematics educators in this important area. The majority of the tasks which relate to Euclidean transformations and which were devised by Piaget are essentially two-dimensional even though they often make use of three-dimensional models. The previously described study was two-dimensional in nature. Likewise the research of Perham, of Schultz, and of Kidder, reported elsewhere in this volume, were restricted to two-dimensional constructs even though three-dimensional materials were sometimes used. Earlier studies by St. Clair (1968), Shah (1969), Williford (1972), and Usiskin (1972) were all teaching studies which focused mainly (if not totally) on two-dimensional isometries.

There have been some exceptions. However, these are found mainly among psychologists interested in spatial mental processes who are not interested specifically in Euclidean transformations or mathematics education. One of the most notable of these is a study by Shepard and Metzler (1971). They presented adult subjects with pairs of computer generated perspective drawings of three-dimensional block configurations. The two drawings in each pair were either identical or very similar to each other, but were rotated with respect to each other. They found that the time which the subjects required to judge the identity of the pairs was a linear function of the size of the rotation. Further, the slope of the obtained function indicated an average rate of mental rotation of about 60 degrees per second.

Other studies include Kraunak and Raskin's (1971) investigation of the influence of stimulus dimensionality, age, and sex, upon perception and discrimination of two and three-dimensional geometric forms. After a training session, 64 children, aged 38 and 54 months, were required to match a two or three-dimensional stimulus. They found that the older children made more correct responses than the younger children on the three-dimensional condition.

Also, Kuezansky, Rebelsky, and Dorman (1971) examined the development of size constancy perception of both two and three-dimensional stimuli in children ages 3 to 6. The children were required to select one of two stimuli with respect to a standard stimulus. The researchers found that errors decrease with age, and more errors were made with two-dimensional than three-dimensional objects.

Finally, Lord (1941) studied the spatial orientation ability of children in the elementary school. Three-hundred-seventeen children were administered a series of four tests: a test of orientation with reference to direction, a test of orientation with reference to cities in space, a test of orientation in the community, and a test of the ability to maintain orientation during travel. The results showed that ability improved as grade increased, and males scored higher than females on all tests but orientation in the community, where performance was equal.

Rationale

Mathematically speaking, there are many instances when three-dimensional arrays and graphs are very useful for consolidating data. Although it is possible to use many one or two-dimensional graphs in place of a three-dimensional one, the results are much less efficient and meaningful. The efficiency of the three-dimensional array is countered, however, by the difficulty in creating, maintaining, and transforming three-dimensional mental images. An interesting hypothesis with regard to this observation is that the ability to conceptualize in three dimensions is related to strategies consistent with formal operational thinking (in the Piagetian sense). That is, the ability to fully utilize the data consolidation capabilities of three-dimensional configurations may be delayed until the advent of formal operations. It is at this time that the child is first able to mentally hold certain variables constant while another is varied. At this stage of development the child is no longer tied to conceptualizing in terms of concrete activity. He is now able to use the conclusions obtained from concrete activity as premises for further deduction. Perhaps it is just such abilities that are required when working with three-dimensional graphs and arrays. The subject must hold one variable constant so that he can mentally visualize how the other two vary under that condition.

This study used a series of tasks requiring the subject to locate one-dimensional configurations of wooden balls embedded in various one, two, and three-dimensional arrays. The tasks were designed in such a way that, from a probabilistic point of view, it would be easier to locate the one-dimensional configurations embedded in the three-dimensional array than in the one or two-dimensional arrays. Even so, it was anticipated that the three-dimensional task might be the most difficult especially for those not yet at the age of formal operations, since these subjects may not yet have the mental structure necessary to formulate an effective search strategy in three dimensions. Further, since the tasks were designed so that the subject was forced to mentally rotate the configurations in order to make the necessary comparisons, an already difficult search task was made even more difficult.

Materials

The arrays were made of blue, yellow, and red wooden spheres which were one inch in diameter and connected by dowels 1.5 inches long. There were four different configurations (see figure 2) and four different arrays: a linear array of 64 balls (1 x 64), a square array of 49 balls (7 x 7), a cubical array of 64 balls (4 x 4 x 4), and a two-dimensional network of arrays laid out as shown in figure 3 (six 4 x 4).

Figure 2. The four configuration types

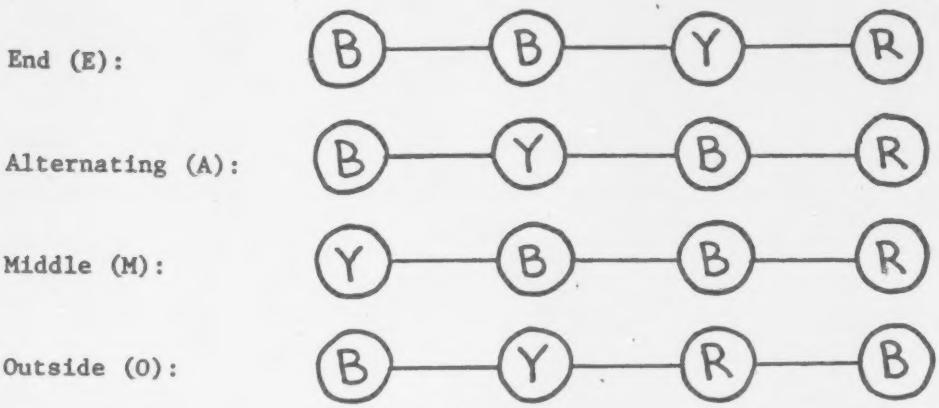
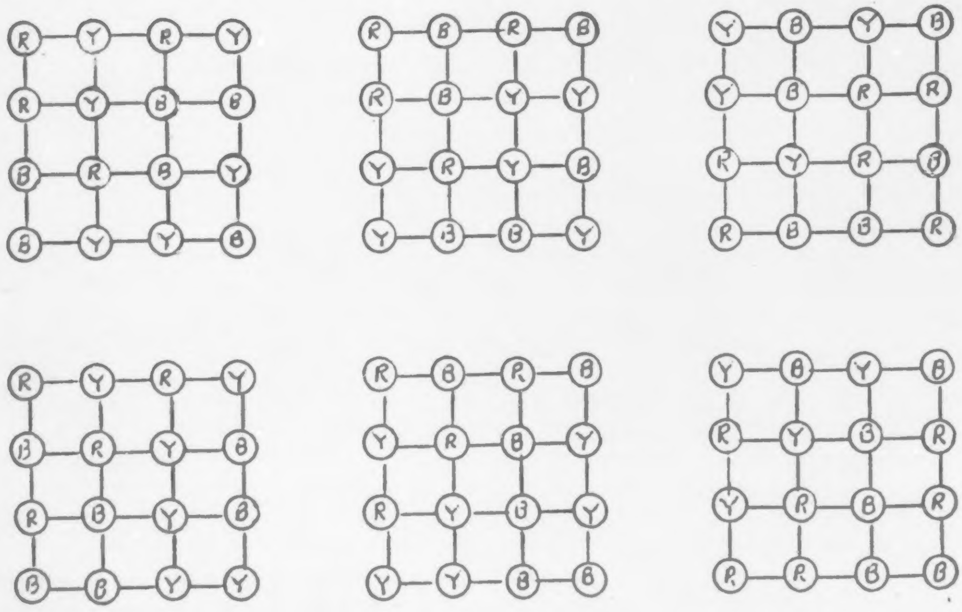

End (E):
B B Y R
Alternating (A):
B Y B R
Middle (M):
Y B B R
Outside (O):
B Y R B

Figure 3. Network of six 4 x 4 arrays


R Y R Y
R Y B B
B R B Y
B Y Y B
R B R B
R B Y Y
Y R Y B
Y B B Y
Y B Y B
Y B R R
R Y R B
R B B R
R Y R Y
B R Y B
R B Y B
B B Y Y
R B R B
Y R B Y
R Y B Y
Y Y B B
Y B Y B
R Y B R
Y R B R
R R B B

The arrays were carefully designed so that there was exactly one configuration embedded in each and so that the total number of embedded 1 x 4 elements was least for the cube, more for the square, and most for the segment. Specifically, the maximum number of 1 x 4 elements each subject needed to check (counting rotations) was 96 in the cube, 112 in the square, and 122 in the segment. Further, for both the segment and the square the subject was confronted with the dual nature of the interior spheres, which could assume either the role of endpoints or interior points of the required configuration. This complication did not exist with the cubical network since any sphere on a face had to be an endpoint of the desired configuration, and any interior point had to be an interior point of the configuration. Hence, from a purely probabilistic and logical point of view, the configuration should have been easiest to find in the cube, harder to find in the square, and most difficult in the segment.

The network of arrays (six 4 x 4) was intended to be most like the three-dimensional array. It had 96 elements to be checked (as did the cube), and, further, it possessed the above mentioned property requiring endpoints and interior points of the network to correspond to endpoints and interior points of the configuration respectively. While it is true that this network was composed of more balls (96 vs. 64), the important variable is not the number of balls, but the number of 1 x 4 embedded elements to be checked. Hence, this fourth network was like the cubical array in all important respects, except that it was two-dimensional rather than three-dimensional.

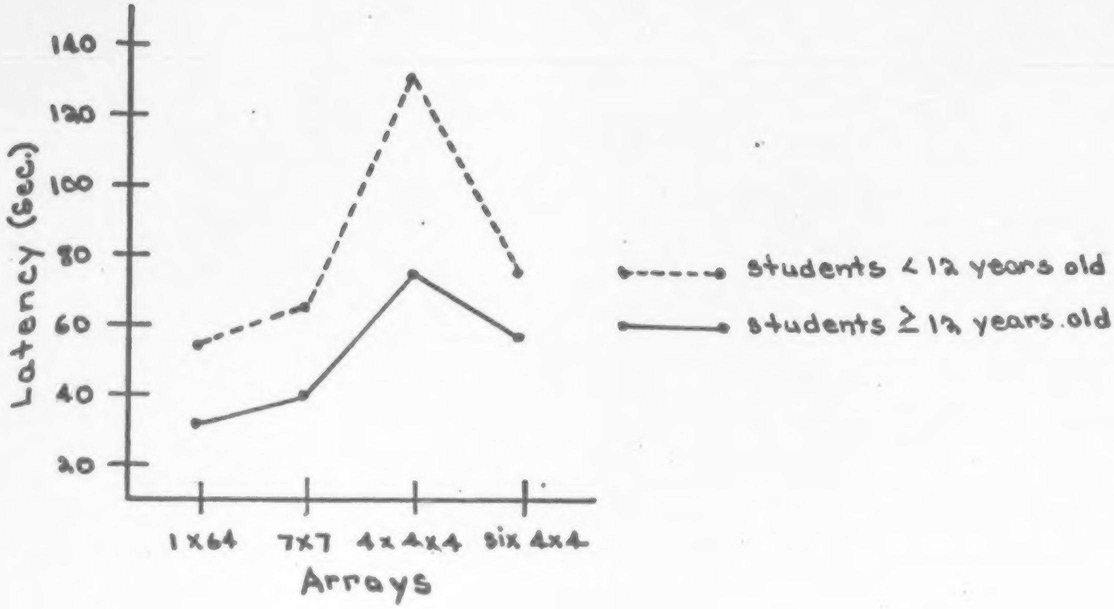
Procedure

Twenty-two fifth graders, 20 seventh graders, 20 ninth graders, and 22 college students were tested during a one-on-one video taped session. One of the four networks was placed before the subject, who was then shown one of the four configurations and asked to find its location in the network before him (wherein it was embedded exactly once) without touching either the configuration or the array. The time to solution (latency) was recorded. Next, he was given a second and then a third configuration to find in the same array. Once again the latency for each was recorded. This same procedure was repeated with the remaining three networks using the same configuration throughout.

Results

The students were divided into two groups according to a roughly acceptable age for the appearance of formal operations: subjects under 12 years in one group and the remaining subjects in the other. The mean latencies for each group were determined across arrays and the profiles recorded graphically (see figure 4). Multivariate techniques were employed using the Biomedical computer program BMDX63 to determine whether the profiles of the two groups were statistically different (Morrison, 1967). The analysis revealed that the group profiles were statistically parallel (indicating no group by array interaction) and that the mean latencies for the groups were significantly different ($F = 5.059$, $p \leq .05$). Further, the profiles were shown to be "non-flat" indicating that the effect due to array was significant (F = 5.081, p .01).

Figure 4. Mean latencies for various tasks



The video tapes were repeatedly and thoroughly reviewed by the experimenters, and many conjectures with regard to problem solving were made (Moyer & Johnson, in press). Since these do not contribute directly to the development of this chapter, they have been omitted.

Conclusions

In both profiles the mean latency for the cubical task was significantly greater than the mean latencies for the remainder of the tasks. This was a clear indication that the cubical task was the most difficult of the tasks. This was not in keeping with what would be predicted by a logical analysis of the tasks involved. One hypothesis regarding the cause of this discrepancy is that the subjects were unable to formulate and/or carry out exhaustive search patterns which were equally effective in all four situations.

The profiles also indicated that the cubical task was more difficult than the equivalent two-dimensional task (six 4 x 4). Hence, the presence of a third dimension must make the task more difficult. However, there was another interesting factor to be noticed in the profiles: the six 4 x 4 task was more difficult than the 7 x 7 task. Here there is no added third dimension to confound the subject. Evidently the fact that center spheres in the array correspond to center spheres in the configuration and end spheres in the array correspond to end spheres in the configuration was not helpful to the subjects. Perhaps it can be hypothesized that if the subject is unable to utilize the above mentioned correspondence, then the fact that the six 4 x 4 array has 96 spheres while the 7 x 7 has only 49 makes the former a more difficult task in spite of the fact that it has fewer 1 x 4 elements (96 as compared to 112).

In general, it was expected that the subjects at or above the age of formal operations (approximately 12 years) would bring cognitive structures to bear that are maximally efficient and are also most in accord with the logical analysis. While it was true that the older subjects performed significantly better than the younger subjects, it cannot be said that their performance was in accord with the logical analysis. Perhaps the method of dichotomization is to blame here. The age of 12 is, at best, a crude approximation to the actual age at which a particular child's thinking becomes formally operational. Perhaps the results would be different if a Piagetian task were used to determine which subjects were at the stage of formal operations.

The Third Study: Advance Organizers and Two-Dimensional Isometries

Johnson (1973) merged Ausubel's theory of advance organizers (1960) with two issues in Piaget's theory of cognitive development: centering and egocentrism. Specifically, the research examined the value of several instructional modes in overcoming certain cognitive deficits which contribute to the discrepancy between children's cognitive structures and the mathematical structure of Euclidean transformations.

Ausubel has been among those educational psychologists who have been concerned with the sequencing of learning experiences. Ausubel (1963) has stated:

> Before we could ever hope effectively to manipulate the classroom learning environment for the optimal acquisition of meaningful subject matter, we would first have to know a great deal more about the organizational and developmental principles whereby human beings acquire and retain stable bodies of knowledge. (p. 3)

Ausubel (1960) suggests that one important variable in the acquisition and retention of knowledge is the existence of clear, stable, and relevant ideational scaffolding in cognitive structure. Another is the discriminability of the new knowledge from knowledge already possessed. He maintains that the acquisition of new concepts can be facilitated if the learner has available to him, in advance, a very general, abstract, and inclusive statement which subsumes the new and more specific knowledge to be learned. Such a statement enables the learner to easily and meaningfully assimilate the new information. This statement is termed an advance organizer.

One of the purposes of this study was to investigate the possibility of using applications, manipulatives, and games as organizers. It seemed that if applications, manipulatives, and games are incorporated in the introductory material (advance organizers), it would aid in overcoming the child's centering and egocentric thinking. For purposes of his study, games were viewed as the interaction between children; using manipulatives was defined as taking real models and formulating mathematical models; and using applications was defined as taking mathematical models and formulating real models.

Subjects

There were 120 fourth graders, selected from two Chicago public schools and one Skokie public school. The schools were similar with respect to location, socio-economic level of parents, and student population.

An investigation showed that neither the curriculum nor the textbooks included the learning material to be used in the study. The teachers indicated that the students had not been exposed to the material in the learning units.

Design

The subjects were divided into three groups: those receiving advance organizers, those receiving post organizers, and a control group which received no organizers. The subjects in each of the two treatment groups were further divided into three subgroups: those working individually, those using manipulatives in groups, and those using applications in groups. Finally, half the subjects in each of these six groups received exactly one organizer while the other half received several (see figure 5).

Figure 5. Design of the study

Advance Organizer						Post Organizer						Control
individual		manip. group		app. group		individual		manip. group		app. group		
one	sev	one	sev	one	sev	one	sev	one	sev	one	sev	

Instruction

The learning task consisted of a self-instructional unit which dealt with Euclidean reflections and rotations. In the unit the transformations were defined and some of their properties were given. Also, examples were given in which one plane figure could be mapped onto another by either a rotation or a reflection. Upon completion of the unit the subjects were expected to be able to recognize if a transformation had taken place.

The organizers were such that the students were asked to perform transformations at an intuitive level. The following are two examples of the organizers used: (1) The students printed and invented words in capital letters which have vertical and/or horizontal symmetry; e.g., BOBO, HEX, HAH, etc.; (2) The students were instructed to write words on transparent screens so that a person on the other side of the screen or standing on his head would be able to read them.

Results

The results showed that the subjects receiving advance organizers scored significantly higher than the subjects receiving post organizers or no organizers. Also, students given several concrete models (or applications) were superior to students given one model (or application) on the decentering sub-examination. Finally, students working in small groups were less cognitively egocentric than students working individually.

Conclusions

This study gives a fine example of an instructional structure which is based both on the cognitive structure of the learner and the mathematical structure of a topic to be taught (transformation geometry, in this case). The study clearly shows that once certain deficits in cognitive structure are identified (egocentrism and centering) it is possible to overcome them by the proper development of an instructional sequence. If these deficits are ignored, the learning is not as efficient or as complete.

Summary

Mathematics is a deceptively rigorous body of knowledge. In most instances the finished, monolithic structure that confronts the student of mathematics belies the process that gave rise to it. Indeed, the last step in the process of mathematical creation is most often the careful rendering of a long series of intuitive hunches and ideas into rigorous, mathematical logic. This uncontrovertible fact in itself should warn mathematics educators to beware of naively creating instructional programs based solely upon the structure of a finished, logically perfect, mathematical body of knowledge. Mathematics is not created in this way so there is no reason to assume that it is best taught in this way.

This chapter has presented a series of three studies which have used Euclidean transformations as a vehicle for investigating the process of determining the nature and/or difficulty of a mathematical task. Isometries were deliberately chosen since rigid motions are encountered by children every day in all situations; yet, until recently, they have seldom been formalized in the classroom. Further, research into children and adults' spatial abilities (especially as they relate to verbal abilities) has encountered renewed interest in the psychological realm, where it traditionally has been the center of interest only for researchers. This renewed emphasis has spilled over into mathematics education circles, where the relationships between spatial and geometric abilities are now of keen interest.

The three studies presented here confirm that there are no ready-made models that can be used in determining the scope and sequence of mathematics curricula. The first two studies confirmed that young children and older students alike have their own way of structuring mathematical concepts, which does not necessarily conform to the way the finished mathematical product is structured. The third study showed that a successful instructional sequence can be designed once the proper information about the cognitive shortcomings of the learner is discerned. However, even then, a theory for doing so is seldom ready-made. A successful method is found only through coordinated, thoughtful, persistent probing.

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Understanding of Frames of Reference by Preservice Teacher Education Students

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The investigation reported in this chapter studied the responses of older students to the Piagetian "water bottle task." The "water bottle task" as used and reported by Piaget and Inhelder (1967) has been administered by many researchers for the purpose of investigating children's understanding and use of a horizontal frame of reference. It has often been assumed that most adults (ages 18 and above for our purposes) would have very little difficulty observing and reporting the constant orientation of the liquid surface in a tilting container to be horizontal. The Genevans' themselves, suggested that very few individuals above the age of 12 years would not have developed the understanding necessary to perform this task correctly. (Piaget & Inhelder, 1967, p. 408)

A pilot study using several classes of university students was carried out to determine whether the above assumption was justified. These students were enrolled in a preservice mathematics content course or in a mathematics methods course for elementary and special education majors. They were asked to complete the following task. Each received a sheet of paper as shown in figure 1. It was explained that the first diagram showed a bottle of water about 1/3 full. The second diagram showed the same bottle tilted approximately 45 degrees. They were asked to draw the water line in the second bottle. To the investigators' surprise, approximately 45 per cent of the water lines drawn deviated enough from the horizontal to be considered incorrect. Clearly, further investigation was warranted to determine the extent of college students' inability to correctly complete this task, to provide some explanation of their inability, and to begin to provide some insight into implications of this inability for learning concepts in geometry.

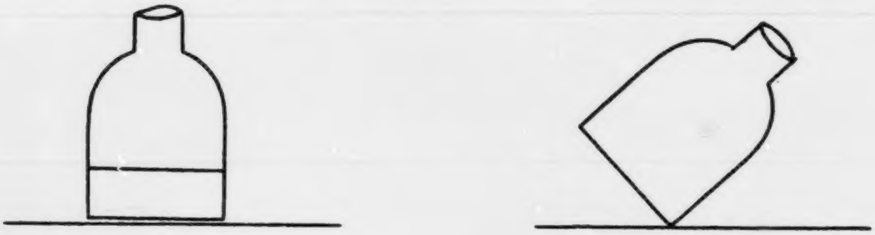


Figure 1. "Bottle Task" Sheet

Review of Literature

A summary and discussion of the work of Piaget, Inhelder and others (1960, 1967) related to the water level task is given before reporting an investigation designed to aid in the determinations listed above. Special emphasis is given in the summary to their interpretation of their observations in terms of frames of reference and coordinate systems.

Frames of Reference

Piaget and Inhelder regard the conceptualization of a frame of reference as fundamental to an individual's ability to deal with the orientation, location and movement of objects; and hence, the very culminating point of the entire psychological development of Euclidean space (1967, p. 416). They characterize a frame of reference, or reference system, as an organization of all positions in three dimensions simultaneously (1967, p. 375). More specifically, "the frame of reference constitutes a Euclidean space after the fashion of a container, relatively independent of the mobile objects contained within it, . . ." (p. 376). The 'axis' within the reference system are objects or positions which remain invariant (actually or hypothetically) under a transformation or action within the system (p. 377). However, more important than the actual choice of stationary reference objects is the "possibility of co-ordinating positions and intervals without limit, through constantly enlarging the original system" (p. 377). Hence, a reference system involves relating mobile objects to actual invariants in the perceptual field. Eventually, these "real" invariants are replaced by more abstract reference points, such as horizontal or vertical axis. A coordinate system is then merely a formalization of a frame of reference.

According to Piaget and Inhelder, the development of an individual's use of reference systems as an aid to the organization and recall of objects about him can be detected by the ability of the individual to utilize the "natural" reference frame, vertical and horizontal.

> "In putting the problem of co-ordinates before the child one is in fact compelled to make reference to the natural axes; namely, horizontal and vertical, since the child himself sooner or later introduces them of his own accord. To find out whether the child has any real understanding of these notions, however, it is necessary to study how he discovers real physical laws in drawing conclusions from his little experiments. That is to say, laws such as the constance of the surface of a liquid whatever the angle of the container,..." (Piaget & Inhelder, 1967, p. 380)

It is the use of a frame of reference which enables the individual to correctly extract certain relationships between objects from what he perceives, such as the orientation of the surface of a liquid.

Piaget and Inhelder observed the use of reference frames when assessing children's ability to recall and record the behavior of the surface of a liquid. They presented children of various ages with two jars, one with straight sides and one which was rounded (no corners or edges). Each jar was 1/4 filled with colored water. Two additional empty jars, one of each shape, were tilted in front of the child. To see if the children used a horizontal reference axis, the children were asked to indicate the water line either with their finger, by drawing it on an outline diagram of the bottles, or by selecting the correct diagram from among several presented.

To observe the use of a vertical reference axis, corks with matchsticks stuck through them were floated on the water in the jar so that the matches were perpendicular to the surface of the water. Alternately, a plumb line was suspended inside of an empty jar. In the first case, the children were asked to draw the position of the "mast" of the "ship" at various orientations of the jar. In the second case, the child was asked to predict the position of the line of string at various jar angles. A third technique was also used. The children were asked to draw a hill and position trees, houses, etc., on it in various places.

Piaget and Inhelder found varying responses to these tasks depending approximately on the age of the child. These responses were categorized into several stages (see figure 2).

Stage III is significant since it marks the start of concrete operations and the discovery of vertical and horizontal reference systems. Although at stage IIIA the responses are marked with trial and error and appear much as in stage IIB, when confronted with the actual experiment (tilting a jar with water in it) children at this later stage appear to discover their errors and learn the correct response. It is significant to note that children before this stage, even when confronted with the experiment, could not see or correct their errors. Finally, at stage IIIB, the response to the tasks is immediately one which indicates that the water line is horizontal. This stage is reached, on the average, at age 9, with a range reported by Piaget and Inhelder of from 6.6 to 12. Similar results were found with the tasks for vertical axis use. The reasons these tasks were so difficult for the children stem directly from the definition of a reference system. Piaget and Inhelder maintain that children at the earlier stages have "...not even an awareness of physical or physiological notions of vertical and horizontal, and for a very simple reason, as these results show. The reason is that perception covers only a very limited field, whereas a system of reference presumes operational co-ordination of several fields, one with another" (1967, p. 416). Not until substage IIB do children begin to compare the water level with the position of another object, indicating an attempt to co-ordinate various aspects of their perceptual field. However, the children at this stage use the jar as a reference point, rather than an immobile element of the system. This incorrect use of an internal reference is the source of their errors.

Task	Stage With Approximate Age Levels		Common Responses	Conceptual Implications
	Pre-Operational	I age 4	Cannot draw water line in either case	
Asked to draw water line on diagram when empty jar or diagram is tilted as shown to them	Observing jar with water does not improve response	IIA age 4 to 5	Water line drawn parallel to base but moved toward mouth in each case (no reference frame used)	Does not learn concept even when demonstrated or explained
------------------ Shown tilted jar containing water - asked to draw water line in diagram		IIB age 5 to 7	Water line drawn using some internal reference (e.g. the sides of the jar)	
	Concrete Operations	IIIA age 7 age 11	Use internal reference as in IIB - see error when shown in second task - then uses internal reference	Learn concept from focused experience
		IIIB age 12 and up	Immediate correct response	Know concept already - have learned concept from previous non-focused experience

Figure 2. Prediction of water level orientation by stage, according to Piaget with underlying conceptual implcations.

At substage IIIA, children first realize that there exists a need for some system of reference external to the water and jar. It is here that they begin to make comparisons with the positions of immobile objects such as the table the jar is on, the floor, etc. It is just this type of comparison with external objects which is needed in order to observe the behavior of water without using the jar as a reference. Finally at substage IIIB the idea of an external reference system is operative and the question of the position of the water level is obvious. "It is horizontal," "It is always straight like the table," "The masts are at right angles to the water," "The plumb line is always vertical," are common responses. "...the distinctive feature of this final substage is the general coordination of all angles and parallels throughout the entire field of objects under consideration." (Piaget & Inhelder, 1967, p. 412)

The above description is only a summary of the results reported in great detail in The Child's Conception of Space, chapter 13. For an in-depth consideration of this experiment, the reader is referred to this chapter.

Coordinate Systems

Is there any relationship between an individual's development of a reference frame and his understanding of formal coordinate systems? In The Child's Conception of Geometry, Piaget, Inhelder, and Szeminska investigated the development of the idea of a coordinate system. Presenting subjects with two sheets of paper, one containing a dot and the other semi-transparent, they asked the subjects to locate a dot on the second sheet in such a location so that when the sheets were stacked on top of one another, the two dots would coincide exactly. They again found a gradual development in these subjects. The youngest subjects of four years old only used a visual estimate to locate the dots. The most advanced used two coordinated measurements to locate them.

The results of this experiment are summarized in figure . What is of importance to this paper is that Piaget et al. claim that the development of this ability to use coordinates in such a situation develops in parallel with the development of systems of reference indicated in The Child's Conception of Space, chapters 13 and 14 (1960, p. 155), i.e. the liquid surface tasks. In Piaget's research both were firmly established by the age of 12, except in the most unusual cases.

A Study of Preservice Teachers

In an attempt to replicate the results of the pilot study referred to earlier, and to determine whether the same interpretation could be given incorrect responses that Piaget and Inhelder gave to theirs, a more formal study was designed and conducted during the Spring semester of 1976.

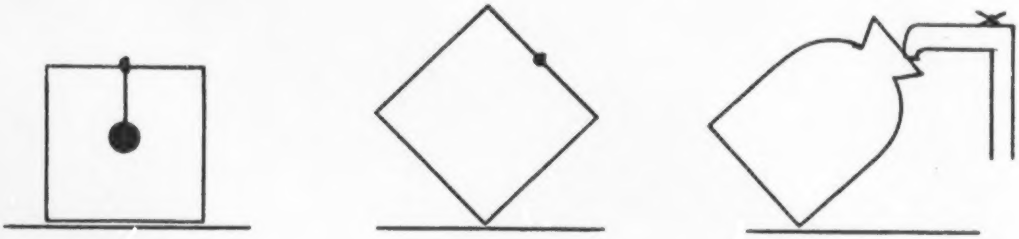
Task	Stage	Common Responses
Asked to locate a point on one sheet of paper in the same location as it is presented on another sheet	I	No use of measurement materials, used visual estimate.
	IIA	Same responses as in stage I except rulers and sticks are used to estimate.
	IIB	Begin to measure but with only one measurement.
	IIIA	Understands the need for two measurements. Uses much trial and error. Often uses one measurement and estimates a second.
	IIIB	Both measurements coordinated. This stage is reached at approximately 9 years old.

Figure 3. Locating a point by stage, according to Piaget, et al. in The Child's Conception of Geometry.

Procedures

The initial set of subjects consisted of 236 elementary education majors, enrolled in either a basic mathematics content course or an elementary mathematics methods course. The 136 students enrolled in the content course ranged in age from 18 to 20 (with a very few older than 20); the 100 students enrolled in the methods course were on the average two years older and most had been exposed to some Piagetian theory in other courses. However, analysis of the data showed no significant differences between the two populations on any parameters of interest to this investigation. Therefore, the results of the study are reported in terms of the combined population of 236 subjects.

The subjects were given three pencil and paper reference system tasks, the "bottle" task (described earlier), the "pendulum" tasks, and the "faucet" task, in that order. In the "pendulum" task a cut-away view of the box was shown, with a weight suspended by a string from the center. The student was asked to draw in the weight and the string in the bottom picture, where he thought it should be if the top box were tilted on a 45 degree angle (figure 4).



"pendulum" task "faucet" task

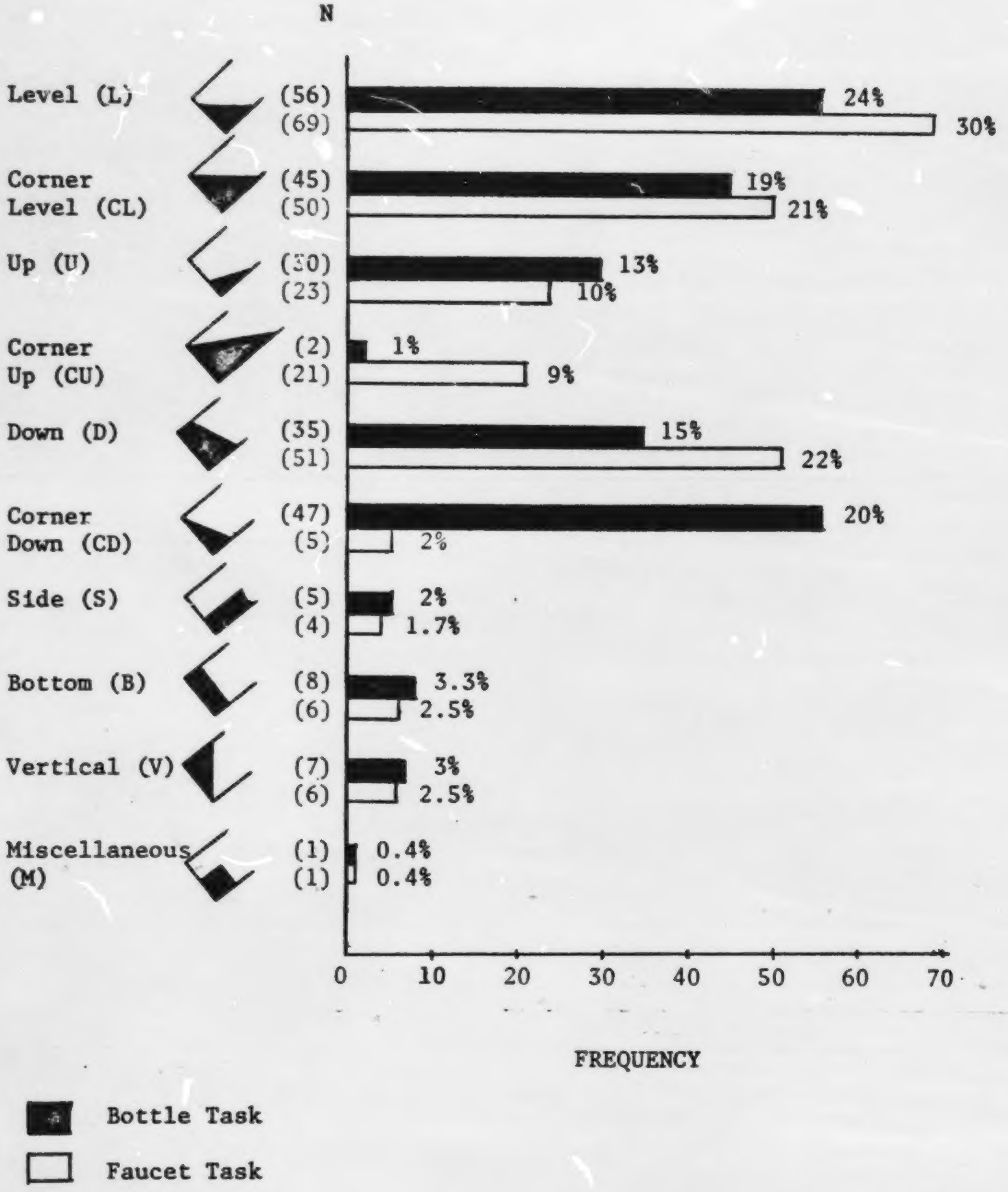
Figure 4

Following the "pendulum" task, the "faucet" task was administered as a check for consistency with the "bottle" task. A cut-away picture of a jug, tilted at a 45 degree angle, was shown being filled by a faucet. The subject was asked to draw in where he thought the water line would be when the jug was half full.

Responses to the "bottle" task and "faucet" task were classified into ten categories for analysis (see Table 1) by the two investigators independently to insure reliability. In the few cases where there was disagreement or when the response was difficult to interpret, students

Table 1

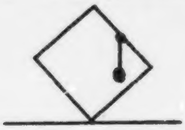
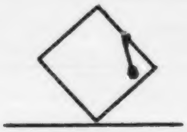
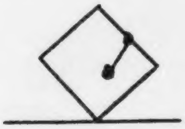
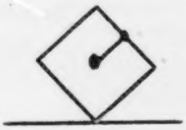
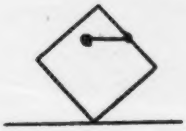
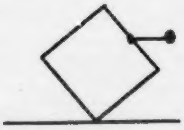
Bottle and Faucet Task Responses*



*The first three major response types were divided into those which were and those which were not drawn from the corner of the bottle diagram. This distinction was considered most important. Since the individuals who drew the water line level, but from the corner, might have drawn it level by chance, using the corner of the bottle as a reference point instead of the table.

Table 2

Pendulum Task Responses

		Frequency	%
Vertical (V)		193	82
Right (R)		31	13
Left (L)		5	2
Side (S)		5	2
Horizontal (H)		1	0.4
Miscellaneous (M)		1	0.4

were either interviewed and asked to explain their response, or the investigators looked at the student's responses on the other two tasks to aid in interpretation. Responses to the "pendulum" task were classified in the same manner into six categories (see Table 2).

The second phase of the study consisted of fifteen minute interviews with 55 subjects, chosen at random from the 236 students in the sample. The purpose of the interviews was to validate the interpretation of the responses on the three reference tasks, and to obtain additional information on the subjects' ability to use vertical and horizontal reference systems. Since both responses that were judged correct and those that were judged incorrect needed validation, all 236 subjects were considered for selection.

The interviews were divided into two parts. In the first part, the subject was asked to repeat the "bottle" task, then to explain why he decided to draw in the water line the way he did. The interviewer noted whether the subject used a rule such as "water must always be parallel to the ground." The subject then repeated the "bottle" task again, but this time he was told that more water was to be added to make the jug approximately two thirds full. The subject again explained his response. Finally the subject was shown the picture of the original jug and asked to select from five pictures of tilted jugs with various water lines drawn in, the one that would look like the original if it were a real jug that had been tilted.

A final experiment was carried out with those subjects who responded incorrectly to the above three versions of the "bottle" task. A gallon size glass jug, one-third full of colored water, was placed on the table in front of the subject. An empty jug of the same size was then held at a 45 degree angle on the table beside the first jug. The subject was asked to hold a ruler against the tilted jug to predict where he thought the water line would be if the first jug was tilted. The first jug was tilted so that the subject could see if his prediction was incorrect. The interviewer then discussed all the subject's responses with him, to see if he could reconcile any inconsistencies and understand why the correct response was, in fact, what actually happens when a jug of water is tilted.

Responses in part one of the interview were classified as either consistent correct (with a rule present), consistent correct (no rule present), consistent incorrect, or inconsistent (see Table 3).

The second part of the interview consisted of the point location problem, a task designed to assess the ability to use a coordinate system. This task was to determine whether the implication made by Piaget and Inhelder that the development of frames of reference and the ability to use coordinate systems develop together is reasonable. A piece of standard typewriting paper was shown to the subject. Four dots were placed on the paper and the subject was asked how he might go

Table 3

"Bottle" Task Responses

Category			Number
Consistent Responses	Correct	Rule Apparent	22
		No Rule Apparent	4
		Total	26
	Incorrect		7
	Total		33
Inconsistent Responses			22
Total Correct			26
Total Incorrect			29
Total Interviewed			55

Table 4

"Bottle" Task and "Faucet" Task Consistency

		"Faucet" Task	
		Correct*	Incorrect
"Bottle" Task	Correct	81	20
	Incorrect	38	97

*Level and Corner Level Responses Both Included

Table 5

Consistency of Incorrect Responses Between the "Bottle" and "Faucet" Tasks

Category Consistent* Incorrect Responses	66
Category Inconsistent Incorrect** Responses	89

*Corner and Non-Corner Responses Pooled

**Correct - Incorrect Response Pairs Included about locating the four dots on another sheet of paper exactly the same size, in exactly the same position. (That is, if one paper was placed over the other, the dots on each sheet would coincide exactly.) A ruler and protractor were available on the table near the subject, but attention was not drawn to them. The interviewer noted if the subject used a reference system in locating the four dots on the second sheet of paper, or just used approximation. If approximation was used, the interviewer erected a screen between himself and the subject and asked how the subject might describe to the interviewer how to locate the four dots if they could not see each other's paper. If this hint did not help, the interviewer asked the subject if he could use the ruler or protractor to simplify the task.

Group Results

The responses to the bottle task were surprisingly varied. Only 43% of the 236 subjects in this part of the experiment drew the water level horizontal, responses L and CL. From the work of Piaget et al, cited earlier, almost 100% correct responses would be anticipated for individuals at this age. It is interesting to note that all the responses found in this study were also reported in the experiments in The Child's Conception of Space. Some of these responses were interpreted by Piaget et al, as pre-operational responses, i.e. below stage III.

The responses to the "faucet" task followed a similar pattern to those in the "bottle" tasks. Here, at most 51% of the responses were judged to be horizontal. All the response categories in the "bottle" task were found in the "faucet" task. Apparently the "faucet" task was somewhat easier for the subjects than the "bottle" task. Tables 4 and 5 compare the responses of subjects on both tasks. It would appear that there exists a substantial inconsistency of responses between the tasks; however, this inconsistency might have been found even between responses given to the identical task administered at different times. With only the data from this phase of the experiment, it is not possible to conclude that the two tasks are not equivalent.

In the "pendulum" task six categories of responses were determined (see Table 2). On this task, 193 or 82% of the subjects responded in category V, the correct response. Of the 43 subjects giving an incorrect response, only 3 gave possible correct responses to both liquid tasks. Table 6 gives a more detailed comparison of these liquid tasks with the incorrect pendulum task.

Table 6

Responses to Liquid Tasks for Subjects Incorrect on "Pendulum" Task

	"Faucet" Correct	"Faucet" Incorrect
"Bottle" Correct	3* (7%)	4 (9%)
"Bottle" Incorrect	10 (23%)	26 (60%)

*Of these 3 subjects, each gave corner-level responses for the "faucet" task.

Interview Results

Since this first phase was given in large group settings, the investigators could not be certain that all subjects understood the tasks or that they were trying to give correct responses. Were all subjects carefully recording their responses or was much of the variation in the data due to poorly drawn responses? How valid were the experimenters' judgements of the response drawings? To answer these questions and investigate possible implications of these results for more common geometric tasks, the interview phase of the study was conducted.

As described earlier, the first part of the interview dealt with the "bottle" task with emphasis on consistency of responses and use of a rule to determine responses. Table 3 summarizes the types of responses found during this portion of the interview.

Twenty-two out of fifty-five students interviewed gave responses which were inconsistent with other responses they gave to the identical task during the same interview. For example, student A responded to the pencil and paper tasks by indicating that the water line should slope downward. He also selected the picture of the jug with the water line sloping downward as the most correct from among the five presented. When asked to show where he expected the water line to be on an actual tilted empty glass jar, he indicated that it should go upward. Of these 22 students, the change in response occurred most often when they were confronted by the actual empty bottle, as in the above example. In every case of inconsistency, the subjects indicated that the water line would be slanted upward, i.e. of category U, regardless of their previous reaction to either the pencil and paper tasks or the selection of response task. When confronted with their inconsistent responses, they were at a loss to explain them. Two individuals did say that they thought the orientation of the water line depended upon the volume of water in the bottle. It is interesting to note that most of the individuals giving incorrect responses were also inconsistent, 22 out of 29.

After the first part of the bottle task interview, the students asked how they decided to draw the water line the way they did. Of the 26 subjects who drew the water line horizontal, 22 had a correct rule such as: "The water line is always horizontal" or "The water line is always parallel to the table." In only four cases was no correct rule apparent.

For those 29 individuals who did not draw the water line horizontal, no one had a precise rule. Many said things such as "As the bottle is tilted, the water gets close to the mouth," or "The water will rise on this side and go down on that side." In every case, the statements made were in terms of the bottle as a reference.

Finally, for those subjects who did not give a correct response to the empty bottle, the bottle of colored water was actually tilted in front of them. Nineteen of these students apparently learned from the demonstrations that the water line was always horizontal. Ten of 29 students could not see the surface of the water as parallel to the table, even when looking at the experiment being performed and after being asked to make the comparison of the surface with the table. A few others admitted that the water surface was parallel to the table only after actual measurement and argument. There is some doubt whether these individuals actually did come to learn the correct response. Perhaps they were just giving in to the interviewer. One student, after apparently learning the correct response, was asked to repeat the original paper and pencil task. Her response was identical to her original response. She had not learned!

The evidence gathered from the interview would suggest that the information gathered in the large group administration of the paper and pencil tasks is reasonably reliable. Where unrealiability exists, it normally is in the direction of interpreting the response of an individual as correct when he actually did not understand the correct response. Only 3 students were found who had given a response judged incorrect to the group tasks and consistently correct responses in the interview.

The interview also provided more insight into the individuals who made incorrect responses. Apparently, even at the age and experience level of the subjects in this study, there exists a rather significant portion who could not learn the correct behavior of the water by watching it.

The results on the second part of the interview, the location of points problem, were quite different from the first part. Fifty of the subjects interviewed measured the distance of each point from each of two perpendicular sides of the first sheet and then located the position on the second sheet by measurements. Four students needed the restatement of the problem as a hint. Only one student still used an approximation of one of the measurements after the hint was given. It was clear to the interviewers that this was a rather easy task for practically all subjects involved.

Since so little difference in the performance of subjects was found on the second interview phase, conclusions concerning any relationship between the tasks cannot be made with confidence. The one student who did use approximation in the location of points task, did have inconsistent responses in the "bottle" tasks section of the interview. She was also among those who did not learn the appropriate response from the actual demonstration of the bottle of water. Evidently, the point location task was substantially easier than the bottle task for the subjects in this study.

Discussion

The results of this investigation are interesting in both their agreement and disagreement with the work of Piaget and Inhelder. The categories of responses and the comments made by the subjects during the interviews were just those reported by Piaget. Those individuals who gave incorrect responses to the liquid tasks explained the movement of the water only in terms of the bottle, whereas those who consistently had the "bottle" task correct explained their responses in terms of some external system of reference. Hence the interpretation of Piaget et al. was quite plausible as an explanation of the observations made in this study.

There are two points of disagreement with the results of similar studies reported in The Child's Conception of Space and The Child's Conception of Geometry. The first difference is that of the ages of the subjects. Piaget et al. indicated that most children have acquired the idea of a natural frame of reference by the age of 9 and, therefore, would have little difficulty with any of the tasks requiring the use of horizontal and vertical references. In contrast, a majority (57%) of subjects in the present study did not perform the tasks correctly. Yet, the subjects in this study were university students mostly between the ages of 18 and 21. The responses of approximately 18% of those students interviewed suggest that not only were their responses incorrect, but that these students were at stage IIB or lower, i.e., at the pre-operational stages. All students interviewed, however, did quite well on the point location task, performing at a level considered by Piaget to be IIIB. In contrast, Piaget et al. in The Child's Conception of Geometry (p. 155) indicated that these abilities develop together. This is the second point of disagreement.

One plausible explanation for these differences stems from the nature of the tasks. The location of points task involves the use of rectangular coordinates to locate positions; and this is a topic in the school mathematics curriculum. The task, as it was presented in the interviews, was not unlike exercises that the students could have experienced in school. Thus, to provide a correct response, a student need only recall these very similar exercises from previous schooling. The liquid and pendulum tasks are not as closely related to formal school instruction. They are, however, similar to experiences these students have every day when drinking or hanging objects. The results of this study suggest that although the concept of formal coordinate systems was evidently acquired, as demonstrated by the successful performances on the point location task, its application to some physical situations was quite difficult since 57% of the subjects could not do the "real world" tasks. These unsuccessful students used a frame of reference only when the situation was obviously mathematical; that is, when it involved mathematical objects such as points. Without such an obvious clue, no relation was seen between coordinate systems and the "real world." Hence, it seems that the difficulty arises with being able to apply geometric concepts in real world situations rather than with the acquisition of the concepts themselves. Additional research is needed to pinpoint the reasons for this lack of coordination between formal and physical geometric concepts.

The lack of coordination between the "real" and formal concepts has serious implications. First, students who had not established this relationship apparently lacked the ability to make comparisons in orientations between real objects and to use natural reference systems. Since Piaget et al. considers the understanding of frames of reference to be fundamental in understanding Euclidean relationships (1967, p. 416), an individual without a working understanding of frames of reference is severely handicapped in obtaining geometric knowledge from his environment.

The fact that the subjects used in this study were teacher education students points to other rather serious implications. A teacher who has not yet coordinated formal geometric concepts with physical situations will not be able to help a student relate these same ideas! Obviously, if the teacher cannot understand the concepts which need to be taught to students, s/he will be even less able to design instruction to teach these concepts. If, indeed, the capabilities of some teacher education students are below these demonstrated minimum levels, research is needed to determine what, if anything, can be done to make these future teachers capable.

Those involved with the teaching of mathematics must be aware that serious problems might exist which could affect the outcome of even the best instruction. They must be alert to the abilities of the students they confront and attempt to adjust instruction to these abilities. As research provides more information, teachers will be able to do a more effective job of adapting instruction to the needs and abilities of their students at all levels.

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Visual Influences of Figure Orientation on Concept Formation in Geometry

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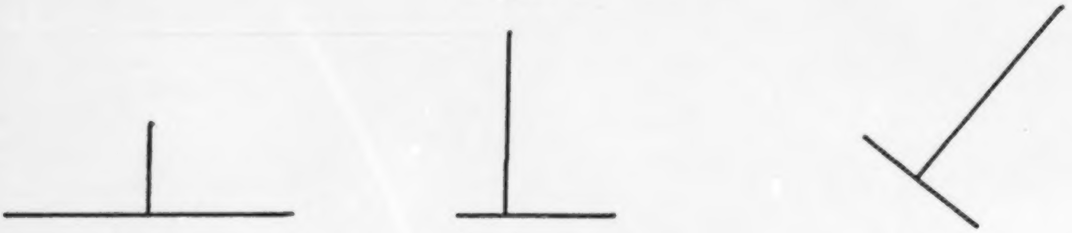
Pictures are important aids for giving students an intuitive grasp of a geometric concept. They provide structures from which students can "read-out" a great deal of spatial information, such as shapes of figures and details within the shapes. For example, school children can easily identify triangles and distinguish right triangles from among other types of triangles. In spite of the great benefits of pictures as instructional aids, they may have some undesirable affects on concept formation.

There are at least two ways that a student may distort a geometric concept because of the influence of pictures. One type of distortion is to make incidental visual clues into essential features of a concept; another type of distortion is to omit properties of a concept that are not visually prominent. As an example of including incidental features in a concept, suppose a student has based his concept of perpendicular lines on figures 1a and 1b.

Figure 1



1a 1b 1c

He may assume that perpendicular lines must be horizontal and vertical and fail to recognize that figure 1c also shows perpendicular lines. As an example of omitting properties of a concept, suppose a student is shown figure 2 as an illustration for a square. Although the student may note that the square has symmetry about a diagonal, he may fail to notice the four right angles and four equal sides. For example, figure 2 looks more like the non-square diamond in figure 3a than the square in figure 3b.

Figure 2

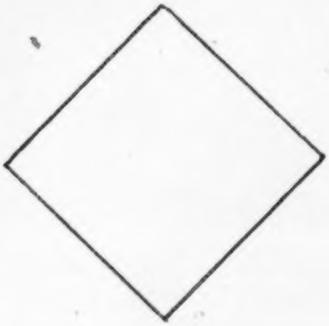
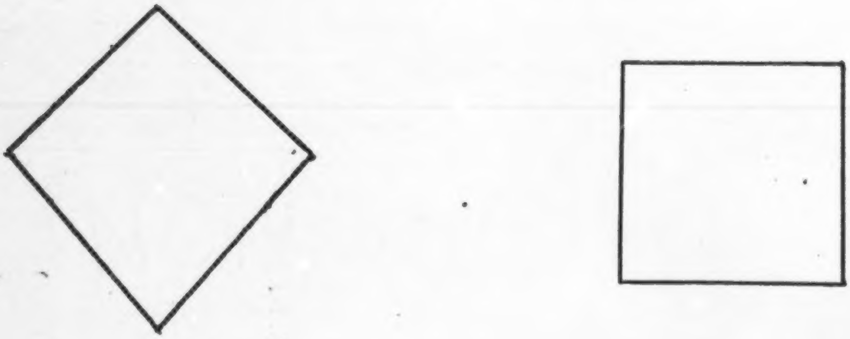


Figure 3



3a

3b

These hypothetical examples of students' forming distorted concepts suggest reasons why students may emphasize certain information in pictures. One explanation is that a student's concept incorporates the visual bias of the illustrations he has seen. This is the view expressed by Zykova (1969) in his studies of the effect of drawings in learning geometry. He concludes that, "If visual experience of a concept is too limited the visual aid that helps pupils master geometric concepts becomes an obstacle in their implementation, since it makes concepts inoperative that are outside the limits of the visual experience" (1969, p. 146). The hypothetical example of a student's assuming all perpendicular lines are horizontal and vertical illustrates Zykova's conclusion.

Another explanation for the formation of distorted concepts focuses on how people code visual information. For example, people tend to stress properties of vertical lines and symmetry about a vertical axis when they are asked to identify and recall figures. A student's self-imposed bias in favor of "upright" figures may be incorporated into the geometric concepts he forms. The hypothetical example of a student's noting symmetry about a diagonal, but missing the regularity of a square, illustrates the perceptual view.

The two explanations of why students form visually biased concepts have conflicting implications for instruction. An implication of the instructional-limitations view is that conceptual biases can be manipulated by changing the bias that is presented in instructional examples; and to prevent a bias from developing, a balanced set of pictures should be presented during instruction. In contrast, the implication of the perceptual-limitations explanation is that a student will impose an "upright" bias on a concept regardless of the examples he is shown.

The present study is concerned with two questions:

(1) Do students form visually biased concepts?

(2) Which view--the instructional-limitations view or the perceptual-limitations view--is the better explanation for why students form visually biased concepts?

To find answers to these questions, the study attempts to induce two types of biases, a bias for upright figures and a bias for tilted figures. Four different instructional formats were designed to provide different experiences with figure orientation. Two provide different types of limited experiences, and two provide different types of varied experiences.

Some of the specific questions that the study investigates are:

(1) Will students form a concept biased in favor of tilted figures if they are shown only tilted figures of a concept?
(2) Will students form a concept biased in favor of upright figures although they are shown as many tilted figures as upright figures of a concept?
(3) Will students learn a concept better if they are shown a balanced set of figures than if they are shown a set of figures with the same orientation?

Review of the Literature

Perceptual Literature

A variety of studies in the perception of shapes and patterns indicate that adult Ss consistently utilize the properties of orientation and/or axial symmetry to identify, classify and group figures. For example, Brown, Hitchcock and Michels (1962) found that Ss had difficulty discriminating an odd figure in a group of five identical figures and one odd figure if all the figures had the same orientation. Zusne and Michels (1962) found that Ss always utilized the property of bilateral symmetry to order a set of figures according to any of the three terms "geometric," "regular," or "familiar." Takala (1951) found that Ss were better at finding a figure embedded in a complex pattern if the figure appeared in the pattern with its axis of symmetry aligned vertically, rather than horizontally. Goldmeier (1937) found that Ss judged a figure with symmetry about a vertical axis to be more similar to a standard figure (which had symmetry about a horizontal axis and a vertical axis) than a figure with symmetry about a horizontal axis.

Subsequent research on the preference for organizing a figure about a vertical axis has attempted to separate the influences of (a) the axis being in a vertical position on the retina (retinal orientation), (b) the axis being in a vertical position in the environment (gravitational orientation), and (c) the axis being conceived (rather than perceived) as vertical (phenomenal orientation). For example, Leaman and Rock (1963) found that Ss preferred symmetry about the phenomenal vertical axis over the retinal or gravitational axis. Steinfeld (1970) found that Ss learned to recognize figures despite changes in the phenomenal axis from training to testing. He suggested that Ss were able to "catch on" to the idea that figures could be tilted and that the top of the figure could be conceived independently of the figure's gravitational or retinal axes.

Attneave (1968) has proposed a visual model that suggests a reason for the preference people have for vertical orientation and vertical symmetry, and also suggests why people are able to recognize that a figure is "tilted." The model consists of a pair of internal Cartesian coordinate systems. One coordinate system describes a figure in terms of local axes in a 3-dimensional space and the other coordinate system describes the tilt of the figural axes from the general axes of the field. A minimal principle governs the orientation of the local axes; those orientations which give the figure simple descriptions, such as arranging the local vertical of a figure with an axis of symmetry, are favored.

A similar model has been proposed by researchers interested in children's discernment of shapes and orientations (Hanfmann, 1933; Braine, 1973). For example, Braine (1973) has suggested that children's judgment of uprightness was determined by a focal feature at the top of the figure and a vertical orientation of the main lines of the figure. The child's internal representation of a shape is in the child's preferred upright orientation for the shape and the representation is tagged with the additional information about the position in which it was presented. An implication of this coding is that, "If the figure presented for copying and its internal representation are not in the same orientation, then the subject may have difficulty reproducing the design in the orientation of the [presented] model " (Braine, 1973, p. 5).

Russian Studies in the Role of Drawings in Learning Geometry

Russian studies in how students use illustrations in learning geometry indicate that under certain conditions mathematical conceptualization resembles the behavior of Ss doing perceptual tasks. The Soviets have been interested in how students attempt to utilize a drawing in solving a geometry problem. To investigate the question, problems were selected that required a student to initially organize a figure with respect to one concept, and then to reorganize the figure with respect to another concept. In general, a reorganization of a figure need not require a reorientation of the figure. However, in many of the Soviet problems the explicit reorganization was a reorientation of the figure. Some parts of the figure formed a configuration that illustrated one concept, and some parts of the figure formed a configuration that illustrated another concept. So, reorientation was required because the inferred upright axes for the two configurations did not coincide.

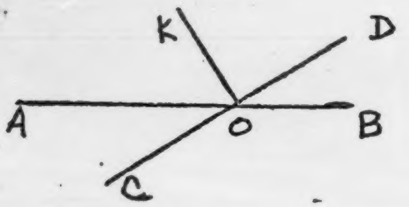
Students tended to view a drawing in terms of a single concept. Zykova noted that,

> (Students regarded visually perceived features as essential features which) gave the concepts an extremely individualized character, and made it difficult (for the students) to work with

> them. Their use was limited to the standard illustrations in the textbook. (Zykova, 1969, p. 95)

The close relationship between the rigid conceptual interpretation students gave to pictures and the inferred orientation of the figure is highlighted by the students' comments. For example, concerning student responses to figure 4, some sample students comments were:

Figure 4



(Students were given vertical angles AOC and DOB and the perpendicular KO to CO and OD.)

> "I didn't understand that angle COK and angle KOD were right angles... I didn't see the drawing like that, the perpendicular is inclined and the right angles can't be seen; then I change the position of the drawing and I saw the right angles." (Zykova, 1969, p. 107)

> "...I didn't understand that it was a right angle (pause) it's not like this (draws a right angle of which one side is horizontal and the other is vertical)." (Zykova, 1969, p. 127)

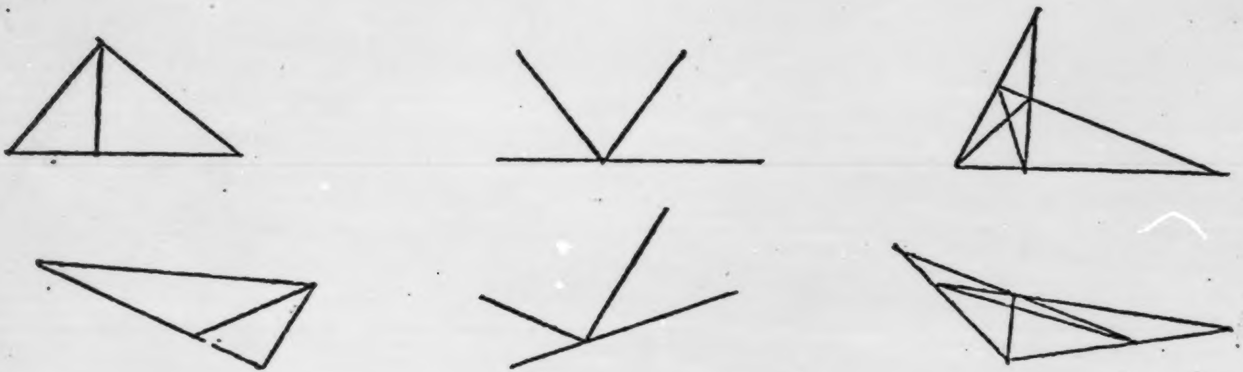
Both the Russian studies and the perceptual literature agree that a student imposes visual limitations on his mental images; the two views differ in their explanations of the cause of the student's biases. The Soviet psychologists have emphasized that instruction limited to standard illustrations accounts for the formation of limited concepts. The perceptual literature emphasizes that individuals tend to organize figures in upright orientation, regardless of how they were presented for study.

Procedure

Concepts Taught

To evaluate the different explanations for visually biased concept formation, and to show that the effects would generalize to different concepts, three different concepts were investigated. The three concepts were "altitude of a triangle," "angle of incidence from a point to a line and its corresponding angle of reflection," and "complete one-point and its three diagonal points." Figure 5 shows illustrations for each of these concepts.

Figure 5



"altitude of a triangle"

"angle of incidence"

"complete 4-point"

Subjects

To investigate whether results would generalize across age levels, the present study included sixth graders, ninth graders, and college students in a pre-calculus mathematics course. Subjects in the study were: 36 sixth grade students at Jefferson Middle School, Madison, Wisconsin; 65 ninth grade algebra students at West High School, Madison, Wisconsin; and 67 college students enrolled in a pre-calculus, algebra, and trigonometry class at the University of Wisconsin, Madison, Wisconsin.

Instructional Materials

At each grade level, students were randomly assigned to a concept, and then randomly assigned to one of four instructional formats. Format A provided experience with only upright figures. Format B provided experience with only non-upright figures. Format C provided an experience of beginning with upright figures and changing to tilted figures, and format D provided the experience of beginning with tilted figures and ending with upright figures.

Instructional booklets consisted of two parts: five pages of illustrations, and a written description of the mathematical concept illustrated by the figures. Figures on the same page were matched for orientation, with the exception of the middle page of three of the formats. A page contained either three or four figures. For each concept, the same set of 16 illustrations was presented in the same sequence for each instructional format.

The expected results from the Soviet view were: (a) instruction limited to one type of figure orientation (upright or tilted) would foster a bias for the orientation presented in instruction, and (b) instruction that included both types of figure orientation would not foster a bias for one type of figure orientation. The expected results from the perceptual view were: (a) in general, students would develop a bias for upright orientation of the illustrations, and (b) instruction that enabled students to relate upright figures to tilted figures would produce the best learning. The author also anticipated that the format containing the upright-to-tilted sequence would produce the best learning.

For each concept, test booklets consisted of three pages. Each page was divided into two sets of figures. Identifications of correct figures had to be made on one set of figures; the other set of figures had to be completed to show a correct illustration of the concept. Instructions for the identification questions were to circle or to shade specified features of the illustrations. The instructions for the figure completion questions were to draw the lines that were needed to complete the figure. No written verbal responses were required.

For each test, half of the identification questions referred to upright figures and half referred to tilted figures. Likewise, the completion questions were equally divided between upright and tilted figures. Students received separate scores for upright figures and tilted figures.

Instructional Sessions

After some preliminary comments, the E explained that each student would receive a pamphlet about a special "geometric figure." Students were told that the pamphlet gave a description of the kind of geometric figure they would learn about and several pages of examples. The E read the following instructions to the students:

(1) The pamphlet is arranged so that the name and the full description of the figure is on a fold-up sheet. Keep the written description in sight while you look at the other pages in the booklet.
(2) On each of the other pages, you'll find several examples of correct figures and some of the features of each figure will be identified in the picture.
(3) There is a "connect-the-dot" exercise at the bottom of each page to help you practice drawing the figures.
(4) Altogether, there are five pages of pictures.
(5) When you study the lesson, try to see how the written description agrees with each picture. Compare the examples on a page to find ways that the pictures are the same and ways that they are different.
(6) The pictures are only a few examples of the possible figures that go with the written description. Try thinking of other possibilities. Remember all the statements must be true.
(7) Use any extra time you have to practice drawing your own pictures.
(8) Go through the pages of the pamphlet in the order of the pages.

It was further explained that after the students had studied their pamphlets, the pamphlets would be collected and new pamphlets would be distributed. The new pamphlets would have questions asking them to identify and draw pictures of the kind of figures they studied.

Testing Sessions

The testing sessions immediately follow the instructional sessions. For both the sixth graders and the college students, testing was done in a large room with all the participants present at the same time. Ninth graders were tested during their scheduled algebra classes and in their customary rooms. Students worked on the test booklets for about 30 minutes.

Results

A grade (including only ninth grade Ss and college Ss) X concept X instructional format repeated measures ANOVA summarized in table 1 indicates that the upright test figures are significantly easier ($p < .001$) than the tilted test figures, the concepts are significantly ($p < .001$) different, and the college students are significantly better ($p < .005$) than the ninth grade students. The interaction between concepts and the orientation of the test figures is significant ($p < .001$). There is no significant difference among the four instructional formats.

Table 2a shows the ninth grade and college means. Table 2b shown the means for upright test figures and tilted test figures (using the scores of ninth grade Ss and college Ss). Each of the means for the concepts "altitude of a triangle" and "angle of incidence" is almost double the mean for the "complete 4-point..." concept. Table 2c shows the means for the three concepts (using the scores of the ninth grade Ss and college Ss). Figure 6 shows the graph for the interaction between concept and orientation of the test figures. The mean for the upright figures exceeds the means for the tilted test figures across the three concepts. The greatest difference (4.58) between the means occurs for the "altitude of a triangle" followed next by the difference (1.42) for the "angle of incidence..." and last by the difference (.07) for the "complete 4-point...."

The means calculated from sixth grade Ss and corresponding to the significant effects for the ninth grade and college Ss are shown in table 3. Table 3a shows the sixth grade mean score. Table 3b shows the means for the upright test figures and the tilted test figures. A one-sided t-test for the means of paired observations does not show a significant difference, at the .05 level ($t=1.4555$), between the upright mean and the tilted mean. However, the ratio (.05) of the difference of the means to the mean score for the sixth grade Ss is similar to the corresponding ratio (.06) derived from the ninth grade and college Ss. Table 3c shows the means for the three concepts. A Neuman-Keuls test for multiple means does not show any significant differences at the .05 level among the three concept means. The order of the means does not correspond to the order for ninth grade and college Ss. The surprising result is the "altitude of a triangle" mean in the lowest position. Figure 7 shows the graph for the orientation of the test figures across the

Table 1

ANOVA Summary Table for Grade x Concept x Instructional Format

Source of Variance	df	SS	MS	F
Grade, 9th and college (G)	1	1,669.5	1,669.5	11.94**
Concept (C)	2	17,376.1	8,688.1	62.15*
Instructional Format (I)	3	283	94.3	.67
G x C	2	567.8	283.9	2.03
G x I	3	263.1	87.7	.63
C x I	6	803.3	133.9	.96
G x C x I	6	674.7	112.5	.80
Ss/Groups (GCI)	96	13,423.8	139.8	
Orientation of Test Figure (O)	1	246	246	20.60*
O x G	1	14.6	14.6	1.23
O x C	2	213.4	106.7	8.97*
O x I	3	67.6	22.5	1.89
O x G x C	2	47	23.5	1.97
O x G x I	3	5.3	1.7	.15
O x C x I	6	57.1	9.5	.80
O x G x C x I	6	166.9	27.8	2.34
O x Ss/group	96	1,146.6	11.9	
Total	239	37,025.8		

* p .001

* p .005

Table 2

Table 2a

Mean Scores for Ninth Grade and College

Ninth Grade	College
32.83	38.12

Table 2b

Mean Scores* for Upright and Tilted Test Figures

Upright Test Figures	Tilted Test Figures
36.48	34.46

Table 2c

Mean Scores for the Three Concepts

Concept	Altitude	Angle of Incidence	Complete 4-Point
Mean Score*	41.49	41.49	23.44

*Ninth grade and college scores were used.

Table 3

Table 3a

Mean Score for Sixth Grade -- 22.73

Table 3b

Mean Scores of Sixth Grade Ss for Upright and Tilted Test Figures

Upright Test Figures	Tilted Test Figures
23.28	22.19

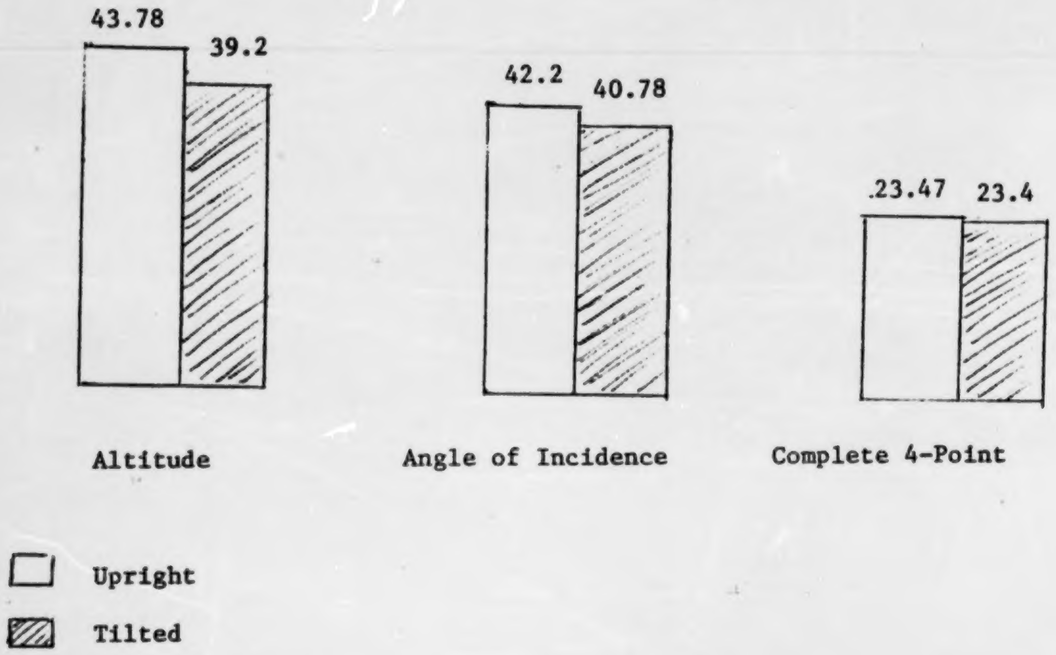
Table 3 (cont'd.)

Table 3c

Mean Scores of Sixth Grade Ss for the Three Concepts

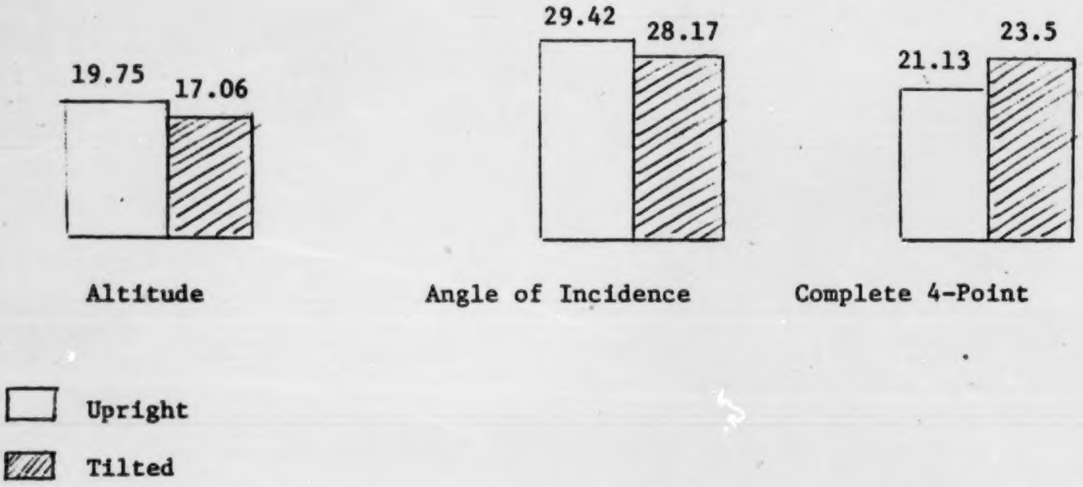
Altitude	Angle of Incidence	Complete 4-Point
18.41	28.79	22.31

Figure 6. Graph of interaction between concept and orientation of test figures*



* Means derived from ninth grade and college scores.

Figure 7. Graph of interaction between concept and orientation of test figures**



** Means derived from sixth grade scores.

three concepts. The parts of the graph for the "altitude of a triangle" and the "angle of incidence..." is similar to the corresponding parts of the graph for ninth grade and college Ss. The mean for the tilted test figures exceeds the mean for the upright test figures for the "complete 4-point...".

Further Observations

In order to identify difficult test figures, answers for individual test figures were examined. For the "altitude of a triangle" concept, obtuse triangles with an exterior altitude were frequently missed. For the "angle of incidence..." concept, pairs of perpendicular lines were frequently missed.

There are two distinct types of figures for a "complete 4-point...". One type of figure has a triangular boundary and the other type of figure has a concave quadrilateral boundary. (See figure 5 for an example of each type.) Ss frequently missed one type of figure. The difficult type varied among the Ss.

In summary, for each concept some Ss placed incorrect restrictions on the shape of figures of the concept, such as requiring that an altitude of a triangle be in the interior of the triangle. The Ss were able to recognize the concept in special cases, but not throughout the full range of examples.

Conclusions

Students do form concepts that are biased in favor of upright figures.

Of the two views proposed to explain why students form visually biased concepts, the perceptual-limitations explanation is consistent with the results of the study, whereas the instructional-limitations explanation is inconsistent with many of the results.

Additional conclusions are:

(1) Students can more easily recognize upright figures for a concept than tilted figures, regardless of instructional experience. The preference develops even when instruction is limited to tilted figures.

(2) Attributes of the figures themselves, rather than the orientation in which the figure is presented, seem to foster the bias for upright figures. The bias for upright figures was strongest for the "altitude of a triangle" concept, less strong for the "angle of incidence..." concept, and weakest for the "complete 4-point..." concept. (In fact, the sixth grade Ss showed a bias in favor of tilted figures of the "complete 4-point..." concept.) The ordering of the concepts may be accounted for by certain features of each set of figures. Figures for the "altitude of a triangle" always contain perpendicular lines. The perpendicular lines may be a clue for mentally aligning the figure to make the perpendicular lines horizontal and vertical. Figures of an "angle of incidence..." always have an implicit axis of symmetry. The axis of symmetry can be mentally aligned in a vertical position to agree with general preferences for a figure's upright position. But the influence of an implicit axis of symmetry as a clue for upright orientation does not seem to be as strong as the clue provided by perpendicular lines. Figures for a "complete 4-point..." lack a common clue for upright orientation. A strong upright bias cannot develop because there are no clues to support it.

(3) Upright biases do not appear to be a reflection of poor conceptualization. For the combined results of the ninth grade and college Ss, there is a direct relationship between a bias for upright figures of a concept and success in learning the concept.

Implications for Instruction

(1) The conclusions of the study suggest that a bias for upright figures may be beneficial for conceptualization. According to some theories of perceptual coding, a figure is most efficiently coded in its preferred upright orientation. New figures that are formed by rotating the upright-prototype figure can be coded with respect to the prototype figure. The new figure is coded as the prototype figure together with the angle of tilt

between the vertical of the prototype figure and the actual vertical of the page. This type of coding implies that the formation of a strong upright-prototype image of a figure should improve recognition of the figure in any orientation. Further investigation is needed to determine ways of helping students form strong upright-prototype images. According to the results of this study, it is neither necessary nor sufficient to show the figure in its upright position.

(2) The presentation of a variety of shapes for a concept is not sufficient to prevent students from forming limited concepts. The visual distinctions that students perceive in figures are often more compelling than the mathematical concept that is illustrated. In consequence, the student forms a concept that is limited to special figures. Further investigation is needed to find ways of helping students form more general concepts.

(3) Further investigation is needed to verify the result for sixth grade Ss that the order of difficulty of the concepts (from easiest to most difficult) is "angle of incidence...," "complete 4-point...," and "altitude of a triangle." This information would be relevant for selecting geometry topics for elementary school. For example, the "angle of incidence..." concept has important physical applications but is rarely taught in elementary school. Providing that children can learn this concept well, it would be desirable to teach it. On the other hand, area formuli, which depend on the altitude concept, are becoming a popular topic in elementary school. If further work confirms that the "altitude of a triangle" is a difficult concept for children, it may be best to postpone the introduction of area formuli until after the sixth grade.

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Research Directions in Geometry

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In the preceding chapters of this monograph you have seen a wide range of studies dealing with space and geometric concepts. The broad areas of (a) young children's view of geometric concepts, (b) children's performance in relation to transformational geometry tasks, (c) the geometries between topology and Euclidean, and (d) the performance of more mature learners in geometric or space situations are included. The authors of these chapters raise many researchable issues related to space and geometry. The prior publication, Space and Geometry: Papers from a Research Workshop, (Martin, 1976) raises additional issues that need to be considered carefully in thinking about future research in space and geometry.

In the paragraphs that follow you will learn of one person's reactions to the results reported in earlier chapters; you will learn of the questions these investigations raised; and you will learn of some of the directions this author sees as important avenues of further research in space and geometry.

The first set of articles deals with, in Lesh's words, "...preoperational concepts or the development of primitive operational concepts." In the article by Lesh on memory improvements one comment strikes me as particularly important:

> (a) The amount of figurative information a child is able to read out of a given situation is determined by the operational system he is able to use. (b) Children will distort what they "see" in order to represent what they understand.

In reflecting on research in learning about space and geometry, Lesh's points justify much of the work that is reported in this monograph. In order to make valid curricular suggestions for geometry in the elementary school, we need to gain baseline data on what children "see" as an adult sees. (No implication is intended here concerning the psychological issue of the nature of perception and/or imagery.) For example, instruction in fractional number concepts is often based upon geometric models in which the child needs to "understand" unit region or segment and equal size parts. If the learner does not operate on these concepts with understanding, that is, if he does not "see" what we see, the learner cannot be expected to gain a stable understanding of fractional number.

But the research dealing with geometric concepts needs to progress beyond a description of what children understand at various age levels; it needs to progress beyond replication and refinement of Piaget's theory of intellectual development; it needs to begin to relate geometric understanding to the type of geometric experiences learners gain in and out of school; it needs to begin to relate geometric understandings to the nature of the instruction used by teachers. If, as Lesh and Mierkiewicz state, "perceiving is not just sensing, but rather the effect of sensori input on the representational system," we need to begin to find effective ways to help youngsters develop representational systems that will permit geometric objects to be perceived as we wish them to be perceived (Martin, 1976).

These views may be considered by some to be heretical or merely foolish to others. However, I would point out that, in the case of geometry, curriculum materials are distinguished by their lack of consistency in grade placement and in sequence of content. Most textbook writing teams appear to use their best judgement regarding content, placement, and sequence, but there appears to be few universally accepted guidelines for inclusion or sequence of geometric material in elementary school mathematics materials. Teachers, therefore, cannot be faulted too severely when they "skip" geometry to spend more time on arithmetic skill development, when there is little agreement among "those in the know." From this writer's view, Piagetian theory provides little guidance in deciding curriculum questions. It does provide information on how children react in certain geometric problem situations. From these data we can identify types of situations that may cause difficulties for some students and some that generally do not. We can also identify activities that we may use to learn what a youngster can do and cannot do. What is needed is a more applicable theory of geometric learning than the oft quoted idea that children's geometric understanding progresses from topological properties to projective properties to Euclidean properties. For as Martin points out,

> On the basis of evidence now available it appears that certain topological concepts such as interior, exterior, and boundary and primitive forms of proximity and separation develop early. Other concepts such as topological equivalence, order, and continuity evidently develop later. Probably some projective concepts also develop early, earlier than many topological concepts. (1976, p. 113)

Fortunately, such a more applicable and testable theory has been proposed by the Van Hiele's. The Van Hiele theory describes various levels of development in geometry, as follows (Wirszup, 1976):

> Level I
>
> This initial level is characterized by the perception of geometric figures in their totality as entities. Figures are judged according to their appearance. The pupils do not see the parts of the figure, nor do they perceive the relationships among components of the figure and among the figures themselves. They cannot even compare figures with common properties with one another. The children who can reason at this level distinguish figures by their shape as a whole. They recognize, for example, a rectangle, a square, and other figures. They conceive of a rectangle, however, as completely different from a square. When a six-year-old is shown what a rhombus, a rectangle, a square, and a parallelogram are, he is capable of reproducing these figures without error on a "geoboard of Gattégno," even in difficult

arrangements. The child can memorize the names of these figures by their shapes alone, but he does not recognize the square as a rhombus, or the rhombus as a parallelogram. To him, these figures are still completely distinct.

Level II

The pupil who has reached the second level begins to discern the components of the figures; he also establishes relationships between individual figures. At this level, he is therefore able to make an analysis of the figures perceived. This takes place in the process (and with the help) of observations, measurements, drawings, and model-making. The properties of the figures are established experimentally; they are described, but not yet formally defined. These properties which the pupil has established serve as a means of recognizing figures. At this stage, the figures act as the bearers of their properties. That a figure is a rectangle means that it has four right angles, that the diagonals are equal, and that the opposite sides are equal. However, these properties are still not connected with one another. For example, the pupil notices that in both the rectangle and the parallelogram of general type the opposite sides are equal to one another, but he does not yet conclude that a rectangle is a parallelogram.

Level III

Students who have reached this level of geometric development establish relations among the properties of a figure and among the figures themselves. At this level there occurs a logical ordering of the properties of a figure and of classes of figures. The pupil is now able to discern the possibility of one property following from another, and the role of definition is clarified. The logical connections among figures and properties of figures are established by definitions. However, at this level the student still does not grasp the meaning of deduction as a whole. The order of logical conclusion is established with the help of the textbook or the teacher. The child himself does not yet understand how it could be possible to modify this order, nor does he see the possibility of constructing the theory proceeding from different premises. He does not yet understand the role of axioms, and cannot yet see the logical connection of statements. At this level deductive methods appear in conjunction with experimentation, thus permitting other properties to be obtained by reasoning from some experimentally obtained properties. At the third level a square is already viewed as a rectangle and a parallelogram.

Level IV

At the fourth level, the students grasp the significance of deduction as a means of constructing and developing all geometric theory. The transition to this level is assisted by the pupils' understanding of the role and the essence of axioms, definitions, and theorems; of the logical structure of a proof; and of the analysis of the logical relationships between concepts and statements.

The students can now see the various possibilities for developing a theory proceeding from various premises. For example, the pupil can now examine the whole system of properties and features of the parallelogram by using the textbook definition of a parallelogram: A parallelogram is a quadrilateral in which the opposite sides are parallel. But he can also construct another system based, say, on the following definition: A parallelogram is a quadrilateral, two opposite sides of which are equal and parallel.

Level V

This level of intellectual development in geometry corresponds to the modern (Hilbertian) standard of rigor. At this level, one attains an abstraction from the concrete nature of objects and from the concrete meaning of the relations connecting these objects. A person at this level develops a theory without making any concrete interpretation. Here geometry acquires a general character and broader applications. For example, several objects, phenomena or conditions serve as "points" and any set of "points" serves as a "figure," and so on. (p. 77-79)

Wirszup continues to note

These levels are inherent in the development of the thought processes....Soviet research has shown that the passage from one level to another is not a spontaneous process concommitant with the students' biological growth and dependent only on his age. The development...proceeds basically under the influence of learning and therefore depends on the content and methods of instruction. (p. 79) The maturation process which leads to a higher level unfolds in a characteristic way; one can distinguish several phases. (This maturation must be considered principally as a process of apprenticeship and not as a ripening on the biological order.) It is then possible and desirable (emphasis added) for the teacher to encourage and hasten it. It is the goal of didactics to ask how these phases are traversed and how to furnish effective help to the student. (p. 82-83)

The phases of development are entitled (a) information, (b) directed orientation, (c) explanation, (d) free orientation, and (e) integration (Wirszup, 1976, p. 83).

> As a result of this fifth phase, the new level of thought is reached. The student arranges a network of relations which connect with the totality of the domain explored. This new domain of thought, which has acquired its own intuition, has been substituted for the earlier domain of thought which possessed an entirely different intuition. (Wirszup, 1976, p. 84)

The Van Hiele levels of thought provide a structure within which a geometric curriculum can be developed throughout the school period. The levels suggest the type of activities we should provide for the learner so that their geometric knowledge will develop. For example, Level I activities would focus in individual figure recognition, production and naming; Level II activities would be focused on determining the properties of individual figures which are now recognized from Level I; in Level III, the relationships between figures and their properties would be emphasized; in Level IV, the knowledge of the previous level would be employed to study geometry from a deductive point of view; Level V activities would be relegated to the collegiate study of geometry by most learners.

It seems that the Van Hiele level theory, taken in conjunction with the Piagetian generated knowledge dealing with specific space and geometric tasks, should generate researchable questions and hypotheses dealing with the learning of geometric knowledge. Wirszup states that the Soviets have carried out extensive studies relating to the Van Hiele level theory. Even so, studies with American learners based upon the levels need to be done. In particular, the Soviet new geometry curriculum begins Level III (semideductive geometry) in grade 4. Certainly this is far in advance of American curriculum in geometry.

Wirszup concludes with the following indictment of our geometry curriculum:

> As a result of unsuccessful experience and convincing evidence the so-called axiomatic methods of initiation into geometry have been recognized by modern educators the world over as unpedagogical. A review of the teaching of geometry in the United States indicates at once that only a very small number of the elementary schools offer any organized studies in visual geometry, and where they are done, they begin with measurements and other concepts which correspond to Levels II and III of thought development in geometry. Since Level I is passed over, the material that is taught even in these schools does not promote any deeper understanding and is soon completely forgotten. Then, in the 10th grade, 15 and 16-year-old youngsters are confronted with geometry for almost the first time in their lives. The whole unknown and complex world of plane and space is given to them

> in a passive axiomatic or psuedo-axiomatic treatment. The majority of our high school students are at the first level of development in geometry, while the course they take demands the fourth level of thought. It is no wonder that high school graduates have hardly any knowledge of geometry, and that this irreparable deficiency haunts them continually later on. (1976, p. 96)

Certainly Wirszup's conclusions in the above citations should not be accepted without empirical evidence. Are the tenth graders at Level 1? Is the "developmental sequence" of the levels accurate? What are the effects of geometric instruction on students' geometric knowledge? Are the levels representative of cognitive functioning in general or are they dependent upon geometric content? How does the "level of thought" theory relate to learning theories? The questions are many, the answers are few at present.

In the discussion of the Soviet curriculum, Wirszup commented:

> It was found that a marked economy in the further study of geometry could be achieved by (and on the basis of) the study of geometric transformations--for example, of axial symmetry--at an earlier time. (This would be the first topic for grade 5....) (1976, p. 91)

Also, Martin's interpretation of Piaget's position led him to make the following assumption.

> Piaget's definition of the nature of knowledge is essentially correct. Knowing requires construction of systems of transformations. These transformations become progressively more nearly isomorphic to transformations of reality. They eventually are combined into systems modeled by the mathematical group. (1976, p. 112)

It seems, then, that "transformations" have a theoretical position as well as a curricular one. Certainly our ultimate goal is for learners to attain a stable view of a Euclidean model of space. Such a goal may be interpreted to imply that the similarity transformations and their invariances need to be mastered explicitly or implicitly by learners. For example, a learner needs to learn that a segment retains its length under an isometry, or that a triangle has a similar figure for its image under a spiral similarity or a dilative reflection. This does not mean, to my mind at least, that a learner must or even may explicitly understand these transformations. Rather it means that the salient characteristics are conserved under motion either physical, representational or conceptual; that is, corresponding segments of similar figures are proportional, corresponding angles equal, etc.

If the quotations from Wirszup and Martin do represent the "truth," then we need to learn more about how youngsters view the effects of transformations. Thomas has provided some information in the present monograph with respect to three isometries. Similarly, Schultz provides information relating attributes of the figure and size (distance between preimage and image) of transformation to the difficulty children found in responding correctly. Kidder found that when student attention was focused on the placement of an image, the student did not always conserve the length of the image figure. Such a result should not be overinterpreted, for when a student is concentrating on the placement of the image, other factors, such as length, may lose his attention.

Even though the work reported in this monograph gives some baseline data, more information is needed. How learners view the effect of isometries and other similarities on all the invariances under these transformations need explication. These studies should be reported completely and systematically so that others can replicate them to confirm or refute the findings. Such results should be compared with "child initiated" transformations. That is, allow a youngster to move a figure in any way he may wish, and ask him about the invariances. If the youngster has consistent results across transformations, then it certainly seems safe to use the formally defined mathematical transformations to help a youngster in his attempt to understand geometric ideas and the nature of space. This, it seems, is what the Soviets have done with some measure of success.

Research on the role of transformations in learning geometry at higher levels is also needed. It seems reasonable to hypothesize that if geometric knowledge is developed by use of transformations, then high school geometry students may be better able to learn about axiomatically organized geometry in a transformation based course. Of course, if Wirszup is correct in his assessment of the level of geometric thought of American tenth graders, then the approach used and its "fit" with the cognitive development may be immaterial.

In all the works in this monograph, you have seen reference made to the learner's cognitive structure. Such reference was made in particular in the Fuson, the Martin, and the Moyer and Johnson articles. Clearly there is no way to measure cognitive structure directly. Rather tasks are given, responses are noted and analyzed, and inferences drawn. The inferences drawn usually are influenced by the theoretical position favored by the author. For example, in the Kidder research, the results could be equally well explained by reference to attention to relevant attributes, as by the position that the author took. Such differences in interpretation will continue to exist as long as there are competing theories. It should also be noted that cognitive structures require time to develop. The amount of time a youngster spends in trying to learn something may be a crucial variable in the study of cognitive structure development.

At least three things are needed so that we can better understand the cognitive structure of a youngster. First we need carefully documented case studies of individual youngsters over a substantial period of time. The basis for these case studies could very well be many of the Piaget inspired tasks. But there need to be others also! In particular, tasks dealing with all the geometric content we hope to see learned eventually should be developed. For example, careful documentation of the learner's behavior as he tries to learn some geometry may be most informative regarding his cognitive structure. Information gathered in case studies would be developmentally and curricularly more useful than that obtained in the usual age sampling techniques.

But we also must admit that longitudinal case studies are difficult to carry out, so we need substantial data gathered by age sampling also. These sets of data for the various concepts and generalizations studied need to be compared to determine if the cognitive structures and developmental "stages" are the same for the two. If they are, more confidence is gained in the data generated by age sampling.

The third activity needed with respect to the longitudinal data is to relate it to the type and amount of instruction taking place in the schools. One of the shortcomings of Piaget's work is that results are not related to educational experience. What are the effects of instruction? Does instruction, or the lack thereof, affect cognitive structure? If so, how?

In this regard, Moyer and Johnson tried to determine the "fit" between a mathematical structure and a learner's cognitive structure. They found discrepancies between the two. Such results should not be unexpected, because the mathematical structure is often imposed for the sake of convenience. As Martin comments,

> "Inherent" sequence in mathematics may actually be only an organizational aid employed by mathematicians....Structure does not automatically determine the sequence of the child's conceptual development. (1976, p. 113)

The major point here is that cognitive structure is a continuously developing entity. Both psychologists and educators would agree to this. Also both groups would agree that an individual's cognitive structure is affected by the environment and interaction with the environment overtime. If the goal is to help a person internalize a particular mathematical structure, then it seems reasonable to first allow the learner to internalize the concepts to be structured. Given this internalized "raw material," the learner can be assisted in developing a structure for it. (Perhaps the two activities go on simultaneously over a period of several years and through a variety of tentative or incomplete "structures.")

This author's major point should now be clear. Development is informative and important, but intervention and instruction based upon developmental understanding is where a major effort needs to be mounted. Some learning occurs incidentally, but mathematical learning depends upon formally organized and planned instruction. If this were not the case, then why did Dietz and Barnett find such large proportions of their college students who responded at low developmental stages with respect to the

Piagetian tasks used. Thus, activities which foster appropriate (with respect to the mathematical material involved) cognitive structures are vitally needed. They should be developed, employed, and evaluated with children at appropriate school levels. Especially important here are the evaluative tools. Tools need to be developed for all the geometric and space concepts which are important. These need to be shared and used widely so that differences in instrumentation do not cloud the results obtained.

Developmentalists and interventionists (perhaps more commonly called psychologists and mathematics educators) need to develop a strong working relationship in order to obtain insight into space and geometry development. The developmentalists need to provide base line information on many geometric-space concepts; they need to do status studies, to investigate developmental theories, etc. The interventionists need to intervene at appropriate places and carefully note the nature of the intervention, the results, and the effects on the learner's cognitive structure. The two groups working closely together sharing information and techniques, sharing conceptualizations and questions, communicating openly and frankly, and replicating often can advance the frontiers of our knowledge about the development of space and geometric concepts and means for effective instruction.

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